Commun. Math. Phys. 184, 411 – 441 (1997)

Communications in
Mathematical Physics

Localization of Classical Waves II: Electromagnetic Waves

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Received: 1 July 1996 / Accepted: 15 August 1996

Abstract: We consider electromagnetic waves in a medium described by a position dependent dielectric constant $\varepsilon(x)$. We assume that $\varepsilon(x)$ is a random perturbation of a periodic function $\varepsilon_0(x)$ and that the periodic Maxwell operator $\mathbf{M}_0 = \nabla^\times \frac{1}{\varepsilon_0(x)} \nabla^\times$ has a gap in the spectrum, where $\nabla^\times \Psi = \nabla \times \Psi$. We prove the existence of localized waves, i.e., finite energy solutions of Maxwell's equations with the property that almost all of the wave's energy remains in a fixed bounded region of space at all times. Localization of electromagnetic waves is a consequence of Anderson localization for the self-adjoint operators $\mathbf{M} = \nabla^\times \frac{1}{\varepsilon(x)} \nabla^\times$. We prove that, in the random medium described by $\varepsilon(x)$, the random operator $\mathbf{M}$ exhibits Anderson localization inside the gap in the spectrum of $\mathbf{M}_0$. This is shown even in situations when the gap is totally filled by the spectrum of the random operator; we can prescribe random environments that ensure localization in almost the whole gap.

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⋆ This author was supported by the U.S. Air Force Grant F49620-94-1-0172.
⋆⋆ This author was supported in part by the NSF Grant DMS-9500720.

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1. Introduction

This is the second of a series of papers on the localization of classical waves. In the first paper we discussed some general aspects of the localization of classical waves, and proved the existence of localized acoustic waves in appropriate random media [FK3]. The present paper is concerned with the localization of electromagnetic waves. This phenomenon arises from coherent multiple scattering and interference, when the scale of the coherent multiple scattering reduces to the wavelength itself. It has numerous potential applications (e.g., [DE, J1, J2, VP, JMW]), for instance, the optical transistor, which explain the recent interest in the localization of light.

Although the localization of light has a lot in common with the localization of acoustic waves, the vector nature of electromagnetic waves poses additional problems for the appropriate arguments, let alone their numerical implementation. (For a discussion of the failure of standard arguments to work for classical waves see [An2].) In this paper we develop adequate tools in order to prove the localization of electromagnetic waves, in a randomly perturbed, lossless periodic dielectric medium with a gap in the spectrum. These tools include interior estimates for the intensity of the electromagnetic field components, properties of an electromagnetic analog of Dirichlet problems in finite domains, bounds on traces of the Green's functions associated with the relevant Maxwell operators, existence of polynomially bounded generalized eigenfunctions, exponential decay of the Green's functions of the underlying periodic medium if the frequency falls in a spectral gap, Wegner-type estimates of the density of states, and more. After all these preparations the proof of localization goes along the same guidelines as in the case of acoustic waves [FK3]. The multiscale analysis developed in [FK3], based on studies of Anderson localization for random Schrödinger operators [FS, FMSS, DK, Sp, CH], is extended to the case of electromagnetic waves, using the new technical tools. As far as the essence of the localization phenomenon is concerned, it remains the same. As in the case of electron waves, a strong enough single defect in a periodic dielectric medium with a spectral gap generates exponentially localized eigenmodes [FK4]. If we have a random array of such defects then, under some natural conditions, the electromagnetic wave tunneling becomes inefficient (that is the main result of the multiscale analysis) and, hence, Anderson localization of electromagnetic waves occurs in spectral gaps of the underlying periodic medium.

To create an environment which would favor localization, one considers first a perfectly periodic dielectric medium (a "photonic crystal", e.g., [JMW]), such that the associated spectrum has band gap structure; the most significant manifestation of coherent multiple scattering is the rise of a gap in the spectrum ("photonic band gaps"). If such a periodic medium with a gap in the spectrum is slightly randomized, eigenvalues with exponentially localized eigenfunctions should arise in the gap. If the disorder is increased further within some limits the localized states can fill the gap completely. *This* *is exactly the medium in which we study electromagnetic waves; we assume an underlying periodic dielectric medium with a gap in the spectrum*. We will slightly randomize such periodic media with a gap in the spectrum and show that, under pretty reasonable hypotheses, Anderson localization occurs in a vicinity of the edges of the gap. (The existence of periodic dielectric media exhibiting gaps in the spectrum has been proved rigorously for $2D$-periodic dielectric structures [FKu1, FKu2].)

We previously considered these questions and media in a lattice approximation, both for classical waves [FK2] and for Schrödinger operators [FK1]. The strategy of this paper and of [FK3] is the same one we used in [FK2], the main differences are of technical nature and due to working on the continuum instead of the lattice. Acoustic waves were similarly treated in [FK3]. Localization created by (non-random) local defects was studied in [FK4].

1.1. Maxwell's equations and localization. In a linear, lossless dielectric medium Maxwell's equations are given by

$$\begin{aligned} \mu\tfrac{\partial}{\partial t}\mathbf{H} &= -\nabla\times\mathbf{E}, & \nabla\cdot\mu\mathbf{H} &= 0,\\ \varepsilon\tfrac{\partial}{\partial t}\mathbf{E} &= \nabla\times\mathbf{H}, & \nabla\cdot\varepsilon\mathbf{E} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{E} = \mathbf{E}(x,t)$ is the electric field, $\mathbf{H} = \mathbf{H}(x,t)$ is the magnetic field, $\varepsilon = \varepsilon(x)$ is the position dependent dielectric constant, and $\mu = \mu(x)$ is the magnetic permeability. We use the Giorgi system of units.

The energy density $\mathcal{E}(x,t) = \mathcal{E}_{\mathbf{H},\mathbf{E}}(x,t)$ and the (conserved) energy $\mathcal{E} = \mathcal{E}_{\mathbf{H},\mathbf{E}}$ of a solution $(\mathbf{H},\mathbf{E})$ of the Maxwell's equations (1) are given by

$$\mathcal{E}(x,t) = \frac{1}{2}\left[\varepsilon(x)|\mathbf{E}(x,t)|^2 + \mu(x)|\mathbf{H}(x,t)|^2\right], \quad \mathcal{E} = \int_{\mathbb{R}^3}\mathcal{E}(x,t)\,dx. \tag{2}$$

Maxwell's equations can be recast as a Schrödinger-like equation (i.e., a first order conservative linear equation):

$$-i\frac{\partial}{\partial t}\Psi_t = \mathbb{M}\Psi_t, \tag{3}$$

with

$$\Psi_t = \begin{pmatrix}\mathbf{H}_t\\ \mathbf{E}_t\end{pmatrix} \in \mathbb{H}, \qquad \mathbb{M} = \begin{bmatrix} 0 & \frac{i}{\mu}\nabla^\times\\ \frac{-i}{\varepsilon}\nabla^\times & 0\end{bmatrix}, \tag{4}$$

where $\mathbb{H} = \mathbb{S}_\mu \oplus \mathbb{S}_\varepsilon$ is the Hilbert space of finite energy solutions; for a given $\varrho = \varrho(x) > 0$, bounded from above and away from 0, we set $\mathbb{S}_\varrho$ to be the closure in $L^2(\mathbb{R}^3;\,\mathbb{C}^3, \varrho(x)dx)$ of the linear subset of functions Ψ with $\varrho\Psi \in C_0^1(\mathbb{R}^3;\,\mathbb{C}^3)$, $\nabla\cdot\varrho\Psi = 0$. The matrix operator $\mathbb{M}$, where $\nabla^\times$ denotes the operator given by $\nabla^\times\Psi = \nabla\times\Psi = \operatorname{curl}\Psi$, has a natural definition as a self-adjoint operator on $\mathbb{H}$. The solution to (3) is then given by $\Psi_t = \mathrm{e}^{it\mathbb{M}}\Psi_0$, it has energy

$$\mathcal{E} = \frac{1}{2}\|\Psi_t\|^2_{\mathbb{H}} = \frac{1}{2}\|\Psi_0\|^2_{\mathbb{H}}. \tag{5}$$

A localized electromagnetic wave can be characterized as a finite energy solution of Maxwell's equations with the property that almost all of the wave's energy remains in a fixed bounded region of space at all times, e.g.,

$$\lim_{R\to\infty} \inf_t \frac{1}{\mathcal{E}} \int_{|x|\le R} \mathcal{E}(x,t)\,dx = 1. \tag{6}$$

If the operator $\mathbb{M}$ has an eigenvalue ω with eigenmode Ψ_ω, i.e., $\mathbb{M}\Psi_\omega = \omega\Psi_\omega$, with $\Psi_\omega \in \mathbb{H}$, $\Psi_\omega \not\equiv 0$, then $\Psi_{\omega,t} = \mathrm{e}^{it\omega}\Psi_\omega$ is a localized electromagnetic wave, i.e., it satisfies (3) and (6). Notice that in this case $-\omega$ is also an eigenvalue of $\mathbb{M}$ with eigenmode $\overline{\Psi}_\omega$, so $\overline{\Psi}_{\omega,t} = \mathrm{e}^{-it\omega}\overline{\Psi}_\omega$ is also a localized wave, since if $\mathbf{J}$ denotes the antiunitary involution corresponding to complex conjugation on $\mathbb{H}$, i.e., $\mathbf{J}\Psi = \overline{\Psi}$, we have $\mathbf{J}\mathbb{M}\mathbf{J} = -\mathbb{M}$. It also follows that the spectrum of $\mathbb{M}$ is symmetric, i.e., $\sigma(\mathbb{M}) = -\sigma(\mathbb{M})$, with $\mathbf{J}\mathbb{M}_+\mathbf{J} = \mathbb{M}_-$, $\mathbb{M}_\pm$ being the positive and negative parts of $\mathbb{M}$. In addition, linear combinations of eigenmodes of $\mathbb{M}$ give raise to localized electromagnetic waves.

If Ψ_t is a solution of Eq. (3), it must satisfy the second order equation $\frac{\partial^2}{\partial t^2}\Psi_t = -\mathbb{M}^2\Psi_t$, so the magnetic and electric fields satisfy the second order equations

$$\frac{\partial^2}{\partial t^2}\mathbf{H}_t = -\frac{1}{\mu}\nabla^\times\frac{1}{\varepsilon}\nabla^\times\mathbf{H}_t, \quad \mathbf{H}_t \in \mathbb{S}_\mu, \tag{7}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{E}_t = -\frac{1}{\varepsilon}\nabla^\times\frac{1}{\mu}\nabla^\times\mathbf{E}_t, \quad \mathbf{E}_t \in \mathbb{S}_\varepsilon. \tag{8}$$

The Maxwell operators $\mathbf{M_H} = \frac{1}{\mu}\nabla^\times\frac{1}{\varepsilon}\nabla^\times$ and $\mathbf{M_E} = \frac{1}{\varepsilon}\nabla^\times\frac{1}{\mu}\nabla^\times$ have natural definitions as nonnegative self-adjoint operators on $\mathbb{S}_\mu$ and $\mathbb{S}_\varepsilon$, respectively. The two Maxwell operators are unitarily equivalent, more precisely

$$\mathbf{M_E} = U\mathbf{M_H}U^*, \tag{9}$$

where $U:\ \mathbb{S}_\mu \to \mathbb{S}_\varepsilon$ is the unitary operator given by

$$U\mathbf{H} = \frac{-i}{\varepsilon}\nabla^\times\mathbf{M_H}^{-\frac{1}{2}}\mathbf{H}, \quad \mathbf{H} \in \operatorname{Ran}\mathbf{M_H}^{\frac{1}{2}}. \tag{10}$$

Thus $\sigma(\mathbb{M}) = \sigma(\mathbf{M_H}^{\frac{1}{2}}) \cup [-\sigma(\mathbf{M_H}^{\frac{1}{2}})]$. We obtain solutions of (3) by setting

$$\Psi_{\pm,t} = \left(\mathrm{e}^{\pm it\mathbf{M_H}^{\frac{1}{2}}}\mathbf{H}_0, \pm U\mathrm{e}^{\pm it\mathbf{M_H}^{\frac{1}{2}}}\mathbf{H}_0\right), \quad \mathbf{H}_0 \in \mathbb{S}_\mu. \tag{11}$$

Conversely, any solution of (3) can be written as a linear combination of at most four solutions of this form.

It follows that to find all eigenvalues and eigenmodes for $\mathbb{M}$, it is necessary and sufficient to find all eigenvalues and eigenmodes for $\mathbf{M_H}$. For if $\mathbf{M_H}\mathbf{H}_{\omega^2} = \omega^2\mathbf{H}_{\omega^2}$, with $\omega > 0$, $\mathbf{H}_{\omega^2} \in \mathbb{S}_\mu$, $\mathbf{H}_{\omega^2} \not\equiv 0$, we have

$$U\mathbf{H}_{\omega^2} = \frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\omega^2} \tag{12}$$

and

$$\mathbb{M}\left(\mathbf{H}_{\omega^2}, \pm\frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\omega^2}\right) = \pm\omega\left(\mathbf{H}_{\omega^2}, \pm\frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\omega^2}\right). \tag{13}$$

Conversely, if $\mathbb{M}(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega}) = \pm\omega(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega})$, with $\omega > 0$, $(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega}) \in \mathbb{H}$, not 0, it follows that $\mathbf{M_H}\mathbf{H}_{\pm\omega} = \omega^2\mathbf{H}_{\pm\omega}$ and $\mathbf{E}_{\pm\omega} = \pm U\mathbf{H}_{\pm\omega} = \pm\frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\pm\omega}$.

Our strategy for proving the existence of localized electromagnetic waves is the following: first the operator $\mathbf{M_H}$ *is shown to have pure point spectrum in some closed interval* $I \subset (0, \infty)$*, with all the corresponding eigenfunctions being exponentially decaying (in the sense of having exponentially decaying local* L^2*-norms). For this operator we prove that the curl of an exponentially decaying eigenfunction is also exponentially decaying, so it follows from (9) and (12) that the operator* $\mathbf{M_E}$ *has also pure point spectrum in the closed interval* I*, with all the corresponding eigenfunctions being exponentially decaying. In addition, it ensues from (13) that the operator* $\mathbb{M}$ *has pure point spectrum in* $\{\omega \in \mathbb{R};\ \ \omega^2 \in I\}$*, with all the corresponding eigenfunctions being exponentially decaying, so the energy densities of the corresponding solutions of (3) are also exponentially decaying, uniformly in the time* t*. If* $\chi_I(\mathbf{M_H})$ *is the corresponding spectral projection, then any solution of (3) given by (11), with* $\mathbf{H}_0$ *in the range of* $\chi_I(\mathbf{M_H})$*, satisfies (6).*

The localization of electromagnetic waves is thus a consequence of Anderson localization for operators $\mathbf{M_H} = \frac{1}{\mu}\nabla^\times \frac{1}{\varepsilon}\nabla^\times$ *on* $\mathbb{S}_\mu$*, i.e., the existence of closed intervals where these operators have pure point spectrum with exponentially decaying eigenfunctions.*

1.2. Statement of results. In this article we study electromagnetic waves in a linear, lossless dielectric medium described by a position dependent dielectric constant $\varepsilon = \varepsilon(x)$. For most dielectric materials of interest, the magnetic permeability $\mu(x)$ is close to one (e.g., [JMW]), so we set $\mu(x) \equiv 1$.

We always assume that $\varepsilon(x)$ *is a measurable real valued function satisfying*

$$0 < \varepsilon_- \le \varepsilon(x) \le \varepsilon_+ < \infty \quad \text{a.e. for some constants } \ \varepsilon_- \ \text{ and } \ \varepsilon_+. \tag{14}$$

Such general conditions on $\varepsilon(x)$, particularly the lack of smoothness, are required on physical grounds. In practice only a few materials are used in the fabrication of periodic and disordered media, in which case $\varepsilon(x)$ takes just a finite number of values, so $\varepsilon(x)$ is piecewise constant, hence discontinuous. The abrupt changes in the medium produce discontinuities in $\varepsilon(x)$, which favor and enhance multiscattering and, hence, localization.

In such a medium electromagnetic waves are described by the formally self-adjoint Maxwell operator

$$\mathbf{M} = \mathbf{M}(\varepsilon) = \mathbf{M_H} = \nabla^\times \frac{1}{\varepsilon}\nabla^\times, \tag{15}$$

acting on the Hilbert space

$$\mathbb{S} = \overline{\{\Psi \in L^2(\mathbb{R}^3;\, \mathbb{C}^3);\ \ \Psi \in C_0^1(\mathbb{R}^3;\, \mathbb{C}^3)\ \text{ with }\ \nabla \cdot \Psi = 0\}}. \tag{16}$$

For the rigorous definition, we start by defining the unrestricted Maxwell operator

$$M = M(\varepsilon) = \nabla^\times \frac{1}{\varepsilon}\nabla^\times, \tag{17}$$

as the nonnegative self-adjoint operator on $L^2(\mathbb{R}^3;\, \mathbb{C}^3)$, uniquely defined by the nonnegative quadratic form given as the closure of

$$\mathcal{M}(\Psi, \Phi) = \langle \nabla \times \Psi, \frac{1}{\varepsilon}\nabla \times \Phi\rangle, \quad \Psi, \Phi \in C_0^1(\mathbb{R}^3;\, \mathbb{C}^3). \tag{18}$$

By Weyl's decomposition (see [BS]), we have

$$L^2(\mathbb{R}^3;\, \mathbb{C}^3) = \mathbb{S} \oplus \mathbb{G}, \tag{19}$$

where $\mathbb{G}$, the space of potential fields, is the closure in $L^2(\mathbb{R}^3;\, \mathbb{C}^3)$ of the linear subset $\{\Psi \in C_0^1(\mathbb{R}^3;\, \mathbb{C}^3);\quad \Psi = \nabla\varphi \quad \text{with} \quad \varphi \in C_0^1(\mathbb{R}^3)\}$. The spaces $\mathbb{S}$ and $\mathbb{G}$ are left invariant by M, with $\mathbb{G} \subset \mathcal{D}(M)$ and $M|_{\mathbb{G}} = 0$. We define $\mathbf{M}$ as the restriction of M to $\mathbb{S}$, i.e., $\mathcal{D}(\mathbf{M}) = \mathcal{D}(M) \cap \mathbb{S}$ and $\mathbf{M} = M|_{\mathcal{D}(M)\cap\mathbb{S}}$. Thus

$$\mathbf{M} = P_{\mathbb{S}} M I_{\mathbb{S}} = M I_{\mathbb{S}}, \tag{20}$$

with $P_{\mathbb{S}}$ the orthogonal projection onto $\mathbb{S}$ and $I_{\mathbb{S}} : \mathbb{S} \to L^2(\mathbb{R}^3;\, \mathbb{C}^3)$ the restriction of the identity map. Notice that $M = \mathbf{M} \oplus 0_{\mathbb{G}}$ and $0 \in \sigma(\mathbf{M})$, so

$$\sigma(\mathbf{M}) = \sigma(M). \tag{21}$$

We can thus work with M to answer questions about the spectrum of $\mathbf{M}$.

In the special case of a homogeneous medium with $\varepsilon(x) \equiv 1$, we will use the notation

$$\Xi = M(1) = (\nabla^{\times})^2, \quad \mathbf{\Xi} = \mathbf{M}(1) = (\nabla^{\times})^2\big|_{\mathcal{D}\left((\nabla^{\times})^2\right)\cap\mathbb{S}}\,. \tag{22}$$

In this article we consider electromagnetic waves in random media obtained by random perturbations of a periodic medium. The properties of the medium are described by the position dependent quantity $\varepsilon(x)$, which we will take to always satisfy the following assumptions.

Assumption 1 (The Random Media). *$\varepsilon_g(x) = \varepsilon_{g,\omega}(x)$ is a random function of the form*

$$\varepsilon_{g,\omega}(x) = \varepsilon_0(x)\gamma_{g,\omega}(x) \ \text{, with } \ \gamma_{g,\omega}(x) = 1 + g\sum_{i\in\mathbb{Z}^3} \omega_i u_i(x), \tag{23}$$

where

(i) *$\varepsilon_0(x)$ is a measurable real valued function which is q-periodic for some $q \in \mathbb{N}$, i.e., $\varepsilon_0(x) = \varepsilon_0(x + qi)$ for all $x \in \mathbb{R}^3$ and $i \in \mathbb{Z}^3$, with*

$$0 < \varepsilon_{0,-} \le \varepsilon_0(x) \le \varepsilon_{0,+} < \infty \ \text{ for a.e. } x \in \mathbb{R}^3, \tag{24}$$

for some constants $\varepsilon_{0,-}$ and $\varepsilon_{0,+}$.

(ii) *$u_i(x) = u(x - i)$ for each $i \in \mathbb{Z}^3$, u being a nonnegative measurable real valued function with compact support, say $u(x) = 0$ if $\|x\|_\infty \le r_u$ for some $r_u < \infty$, such that*

$$0 < U_- \le U(x) \equiv \sum_{i\in\mathbb{Z}^3} u_i(x) \le U_+ < \infty \ \text{ for a.e. } x \in \mathbb{R}^3, \tag{25}$$

for some constants U_- and U_+.

(iii) *$\omega = \{\omega_i;\, i \in \mathbb{Z}^3\}$ is a family of independent, identically distributed random variables taking values in the interval $[-1, 1]$, whose common probability distribution μ has a bounded density $\rho > 0$ a.e. in $[-1, 1]$.*

(iv) *g, satisfying $0 \le g < \frac{1}{U_+}$, is the disorder parameter.*

For electromagnetic waves $\varepsilon_{g,\omega}(x)$ is the random position dependent dielectric constant of the medium. Notice that Assumption 1 implies that each $\varepsilon_{g,\omega}$ satisfies (14) with

$$\varepsilon_{\pm} = \varepsilon_{g,\pm} = \varepsilon_{0,\pm}(1 \pm gU_{+}). \tag{26}$$

For later use we set

$$\delta_{\pm}(g) = \frac{U_{\pm}}{1 \mp gU_{+}}, \quad \text{with} \quad 0 \leq g < \frac{1}{U_{+}}. \tag{27}$$

The periodic operators associated with the coefficient $\varepsilon_0(x)$ will carry the subscript 0, i.e., $M_0 = M(\varepsilon_0)$, $\mathbf{M}_0 = \mathbf{M}(\varepsilon_0)$. We will study the random operators (see [FK3, Appendix A] for the definition)

$$M_g = M_{g,\omega} = M(\varepsilon_{g,\omega}); \quad \mathbf{M}_g = \mathbf{M}_{g,\omega} = \mathbf{M}(\varepsilon_{g,\omega}). \tag{28}$$

It is a consequence of ergodicity (measurability follows from [FK3, Theorem 38]) that there exists a nonrandom set Σ_g, such that $\sigma(\mathbf{M}_{g,\omega}) = \sigma(M_{g,\omega}) = \Sigma_g$ with probability one. In addition, the decompositions of $\sigma(\mathbf{M}_{g,\omega})$ and $\sigma(M_{g,\omega})$ into pure point spectrum, absolutely continuous spectrum and singular continuous spectrum are also independent of the choice of ω with probability one [KM1, PF].

In this article we are interested in the phenomenon of localization. According to the philosophy of Anderson localization we will assume that the operator $\mathbf{M}_0$ has at least one gap in the spectrum.

Assumption 2 (The gap in the spectrum). *There is a gap in the spectrum of the operator $\mathbf{M}_0$. More precisely, there exist $0 \leq \hat{a}a \leq b \leq \hat{b}$ such that*

$$\sigma(\mathbf{M}_0) \bigcap [\hat{a}, \hat{b}] = [\hat{a}, a] \bigcup [b, \hat{b}],$$

so the interval (a, b) is a gap in $\sigma(\mathbf{M}_0)$.

The following theorem gives information on the location of Σ_g, the (nonrandom) spectrum of the random Maxwell operator $\mathbf{M}_g$.

Theorem 3 (Location of the spectrum). *Let the random operator $\mathbf{M}_g$ defined by (28) satisfy Assumptions 1 and 2. There exists g_0, with*

$$\frac{1}{U_{+}} \left(1 - \left(\frac{a}{b}\right)^{\frac{1}{2}}\right) \leq g_0 \leq \frac{1}{U_{+}} \min \left\{1, \left(\left(\frac{b}{a}\right)^{\frac{U_{+}}{2U_{-}}} - 1\right)\right\}, \tag{29}$$

and strictly increasing, Lipschitz continuous real valued functions $a(g)$ and $-b(g)$ on the interval $[0, \frac{1}{U_{+}})$, with $a(0) = a$, $b(0) = b$ and $a(g) \leq b(g)$, such that:

(i)

$$\Sigma_g \bigcap [\hat{a}, \hat{b}] = [\hat{a}, a(g)] \bigcup [b(g), \hat{b}] \ . \tag{30}$$

(ii) *For $g < g_0$, we have $a(g) < b(g)$ and $(a(g), b(g))$ is a gap in the spectrum of the random operator $\mathbf{M}_g$, located inside the gap (a, b) of the unperturbed periodic operator $\mathbf{M}_0$. Moreover, we have*

$$a \leq a(1 + gU_{+})^{\frac{U_{-}}{U_{+}}} \leq a(g) \leq \frac{a}{1 - gU_{+}} \tag{31}$$

and

$$b(1 - gU_{+}) \leq b(g) \leq \frac{b}{(1 + gU_{+})^{\frac{U_{-}}{U_{+}}}} \leq b. \tag{32}$$

(iii) If $g_0 < \frac{1}{U_+}$*, we have* $a(g) = b(g)$ *for all* $g \in [g_0, \frac{1}{U_+})$*, and the random operator* $\mathbf{M}_g$ *has no gap inside the gap* (a, b) *of the unperturbed periodic operator* A_0*, i.e.,* $[\hat{a}, \hat{b}] \subset \Sigma_g$.

Definition 4 (Exponential localization). *We say that the random operator* $\mathbf{M}_g$ *exhibits* localization *in an interval* $I \subset \Sigma_g$*, if* $\mathbf{M}_g$ *has* only *pure point spectrum in* I *with probability one. We have* exponential localization *in* I *if we have localization and, with probability one, all the eigenfunctions corresponding to eigenvalues in* I *are exponentially decaying (in the sense of having exponentially decaying local* L^2*-norms).*

Remark 5. The curls of exponentially decaying eigenfunctions of $\mathbf{M}_g$ always have exponentially decaying local L^2-norms (Corollary 14). Thus the corresponding energy densities (see (2)) also have exponentially decaying local L^2-norms, uniformly in the time t.

Our main results show that random perturbations create exponentially localized eigenfunctions near the edges of the gap. Our method requires low probability of extremal values for the random variables; the following two theorems achieve this in different ways. The results are formulated for the left edge of the gap, with similar results holding at the right edge.

Theorem 6 (Localization at the edge). *Let the random operator* $\mathbf{M}_g$ *defined by (28) satisfy Assumptions 1 and 2, with*

$$\mu\{(1-\gamma, 1]\} \leq K\gamma^\eta \ \text{ for } \ 0 \leq \gamma \leq 1, \tag{33}$$

where $K < \infty$ *and* $\eta > 3$*. For any* $g < g_0$ *there exists* $\delta(g) > 0$*, depending only on the constants* $g,\ q,\ \varepsilon_{0,\pm},\ U_\pm,\ r_u,\ K,\ \eta$*, an upper bound on* $\|\rho\|_\infty$*, and on* a, b*, such that the random operator* $\mathbf{M}_g$ *exhibits exponential localization in the interval* $[a(g) - \delta(g), a(g)]$.

Theorem 7 (Localization in a specified interval). *Let the random operator* $\mathbf{M}_g$ *defined by (28) satisfy Assumptions 1 and 2. For any* $g < g_0$*, given* $a < a_1 < a_2 < a(g)$*, with* $a(g) - a_1 \leq b(g) - a(g)$*, there exists* $p_1 > 0$*, depending only on the constants* $g,\ q,\ \varepsilon_{0,\pm},\ U_\pm,\ r_u,\ a$*, an upper bound on* $\|\rho\|_\infty$ *and on the given* a_1, a_2*, such that if*

$$\mu\left(\left(\frac{g_1}{g}, 1\right]\right) < p_1, \tag{34}$$

where g_1 *is defined by* $a(g_1) = a_1$*, the random operator* $\mathbf{M}_g$ *exhibits exponential localization in the interval* $[a_2, a(g)]$.

Theorems 6 and 7 can be extended to the situation when the gap is totally filled by the spectrum of the random operator, we then establish the existence of an interval (inside the original gap) where the random Maxwell operator exhibits exponential localization. Notice that the extension of Theorem 7 says that we can arrange for localization in as much of the gap as we want.

Theorem 8 (Localization at the meeting of the edges). *Let the random operator* $\mathbf{M}_g$ *defined by (28) satisfy Assumptions 1 and 2, with*

$$\mu\{(1-\gamma, 1]\},\ \ \mu\{[-1, -1+\gamma)\} \leq K\gamma^\eta \ \text{ for } \ 0 \leq \gamma \leq 1, \tag{35}$$

where $K < \infty$ and $\eta > 3$. Suppose $g_0 < \frac{1}{U_+}$ (e.g., if $\left(\frac{b}{a}\right)^{\frac{U_+}{2U_-}} < 2$), so the random operator $\mathbf{M}_g$ has no gap inside (a, b) for $g \in [g_0, \frac{1}{U_+})$. Then there exist $0 < \epsilon < \frac{1}{U_+} - g_0$ and $\delta > 0$, depending only on the constants q, $\varepsilon_{0,\pm}$, $U_\pm$, r_u, K, η, an upper bound on $\|\rho\|_\infty$, and on a, b, such that the random operator $\mathbf{M}_g$ exhibits exponential localization in the interval $[a(g_0) - \delta, a(g_0) + \delta]$ for all $g_0 \leq g < g_0 + \epsilon$.

Theorem 9 (Localization in a specified interval in the closed gap). *Let the random operator $\mathbf{M}_g$ defined by (28) satisfy Assumptions 1 and 2. Suppose $g_0 < \frac{1}{U_+}$ (e.g., if $\left(\frac{b}{a}\right)^{\frac{U_+}{2U_-}} < 2$), so the random operator $\mathbf{M}_g$ has no gap inside (a, b) for $g \in [g_0, \frac{1}{U_+})$. Let $a < a_1 < a_2 < a(g_0) = b(g_0) < b_2 < b_1 < b$ be given. For any $g \in [g_0, \frac{1}{U_+})$ there exist $p_1, p_2 > 0$, depending only on the constants g, q, $\varepsilon_{0,\pm}$, $U_\pm$, r_u, a, b, an upper bound on $\|\rho\|_\infty$ and on the given a_1, a_2, b_1, b_2, such that if*

$$\mu\left(\left(\frac{g_1}{g}, 1\right]\right) < p_1, \quad \mu\left(\left[-1, -\frac{g_2}{g}\right)\right) < p_2, \tag{36}$$

where g_1 and g_2 are defined by $a(g_1) = a_1$ and $b(g_2) = b_1$ (notice $0 < g_1,\ g_2 < g_0 \leq g$), the random operator $\mathbf{M}_g$ exhibits exponential localization in the interval $[a_2, b_2]$.

Theorems 8 and 9 are proved exactly as Theorems 6 and 7, respectively, taking into account both edges of the gap.

Remark 10. Theorems 6 and 8 should be true without the extra hypotheses (33) and (35). They are used in conjunction with a Combes-Thomas argument to obtain the starting hypothesis for the multiscale analysis, in the proof of localization. One may expect estimates similar to Lifshitz tails (e.g., [PF]) for the density of states inside the gap, which would replace (33) and (35) in the proofs. This is how the starting hypothesis is obtained for random Schrödinger operators at the bottom of the spectrum [HM].

Combes and Hislop have announced an improved Combes-Thomas argument inside a gap; they obtain a decay rate proportional to the square root of the product of the distances to the edges of the gap. With this result we would only need $\eta > \frac{3}{2}$ in Theorem 6, but we would still need to require $\eta > 3$ in Theorem 8.

Theorem 3 is proved in Sect. 4; the proof requires periodic operators and periodic boundary condition, studied in Sect. 3. Theorems 6 and 7 are proved in Sect. 7 by multiscale analyses. Dirichlet boundary condition, used in the proofs, is discussed in Sect. 5. The required Wegner-type estimate is in Sect. 6. The starting hypotheses are proved first for finite volume Maxwell operators with periodic boundary condition, using a Combes-Thomas argument for operators with periodic boundary condition (Subsect. 3.2) and Theorem 3. We collect properties of Maxwell operators needed for the proof of localization in Sect. 2, they include an interior estimate for curls and existence of polynomially bounded generalized eigenfunctions.

1.3. Notation. We adopt the following definitions and notations:

- For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we let $|x|_p = (x_1^p + x_2^p + x_3^p)^{1/p}$ for $1 \leq p < \infty$, and $|x|_\infty = \max_{1 \leq j \leq 3} |x_j|$. We set $|x| = |x|_2$ and $\|x\| = |x|_\infty$.
- $\Lambda_L(x) = \{y \in \mathbb{R}^3;\ \|y - x\| < \frac{L}{2}\}$ is the (open) cube of side L centered at $x \in \mathbb{R}^3$; $\overline{\Lambda}_L(x)$ is the closed cube, and $\check{\Lambda}_L(x) = \{y \in \mathbb{R}^3;\ -\frac{L}{2} \leq y_i - x_i < \frac{L}{2},\ i = 1, 2, 3\}$ the half-open/half-closed cube.

- χ_Λ is the characteristic function of the set Λ; we write $\chi_{x,L} = \chi_{\Lambda_L(x)}$.
- A function f on $\mathbb{R}^3$ is called q-periodic for some $q > 0$ if $f(x + qi) = f(x)$ for all $x \in \mathbb{R}^3$ and $i \in \mathbb{Z}^3$.
- A domain Ω is an open connected subset of $\mathbb{R}^3$; its boundary is denoted by $\partial\Omega$.
- $L^p(\Omega; \mathbb{C}^d)$ is the space of $\mathbb{C}^d$ measurable functions $u : \Omega \to \mathbb{C}^d$ with the norm $\|u\|_p = \|u\|_{p,\Omega} = \left[\int_\Omega |u(x)|^p \, dx\right]^{1/p}$. We will often use the space $L^2(\Omega; \mathbb{C}^d)$ and in this case we will write $\|u\|_\Omega$ for $\|u\|_{2,\Omega}$. If $\Omega = \mathbb{R}^d$ we may omit it from the subscript. We write $L^p(\Omega)$ if $d = 1$.
- $C^n(\Omega; \mathbb{C}^d)$ is the linear space of n-times continuously differentiable functions $u : \Omega \to \mathbb{C}^d$, $C_0^n(\Omega; \mathbb{C}^d)$ is the subspace of functions with compact support. We write $C^n(\Omega)$ if $d = 1$.
- The domain, spectrum and adjoint of a linear operator A are denoted by $\mathcal{D}(A)$, $\sigma(A)$ and A^*, respectively .
- If $\mathcal{A}$ is the quadratic form associated with an operator A, its domain will be denoted by either $\mathcal{Q}(\mathcal{A})$ or $\mathcal{Q}(A)$. We also write $\mathcal{A}[\Psi]$ for $\mathcal{A}(\Psi, \Psi)$.
- $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the Banach space of bounded operators from the normed space $\mathcal{X}$ to the normed space $\mathcal{Y}$; $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$.
- For a complex number z its conjugate is denoted by z^*.

2. Properties of Maxwell Operators

2.1. An interior estimate. Let us consider the first order linear differential operator $D = \{D_{\alpha,\beta}\}_{\alpha,\beta=1,\dots,\nu}$, with each $D_{\alpha,\beta} = a_{\alpha,\beta} \cdot \nabla$ for some fixed $a_{\alpha,\beta} \in \mathbb{C}^d$. D is a closed densely defined operator on $L^2(\mathbb{R}^d; \mathbb{C}^\nu)$, whose domain, $\mathcal{D}(D)$, is the closure of $C_0^\infty(\mathbb{R}^d; \mathbb{C}^\nu)$ in the norm $\left(\|\Psi\|_2^2 + \|D\Psi\|_2^2\right)^{\frac{1}{2}}$.

Given an open set $\Omega \subset \mathbb{R}^d$, we define D_Ω as the closed densely defined operator on $L^2(\Omega; \mathbb{C}^\nu)$, defined in the obvious way for $\Psi \in C^\infty(\Omega; \mathbb{C}^\nu)$ with $\{a_{\alpha,\beta} \cdot \nabla\}_{\alpha,\beta=1,\dots,\nu} \Psi \in L^2(\Omega; \mathbb{C}^\nu)$. If $\Omega' \subset \Omega$, it is easy to see that if $u \in \mathcal{D}(D_\Omega)$, then $u|_{\Omega'} \in \mathcal{D}(D_{\Omega'})$ with $D_{\Omega'} u|_{\Omega'} = (D_\Omega u)\,|_{\Omega'}$, so we will simply write Du to denote the function $D_\Omega u$.

Let $\Gamma = \Gamma(x)$ be a measurable function on $\mathbb{R}^d$ whose values are $\nu \times \nu$ complex matrices with

$$0 \le \Gamma(x) \le \Gamma_+ I_\nu \quad \text{a.e. for some constant } \ \Gamma_+ < \infty, \tag{37}$$

I_ν being the $\nu \times \nu$ identity matrix. We say that a function $u \in \mathcal{D}(D_\Omega)$ is a weak solution for the equation $D^* \Gamma D u = f$ in Ω, where $f \in L^2(\Omega; \mathbb{C}^\nu)$, if

$$\langle D\Psi, \Gamma D u \rangle_\Omega = \langle \Psi, f \rangle_\Omega \quad \text{for all } \ \Psi \in C_0^\infty(\Omega; \mathbb{C}^\nu). \tag{38}$$

Theorem 11. *Let D and Γ be as above. For any $\delta > 0$ there exists a constant $\xi_\delta = \xi(d, \nu, \{a_{\alpha,\beta}\}_{\alpha,\beta=1,\dots,\nu}, \delta) < \infty$, depending only on the indicated parameters, such that if $u \in \mathcal{D}(D_\Omega)$ is a weak solution for the equation $D^*\Gamma D u = f$ in an open subset Ω of $\mathbb{R}^d$, with $f \in L^2(\Omega; \mathbb{C}^\nu)$, we have*

$$\langle Du, \Gamma D u \rangle_{\Omega'} \le \xi_\delta \left[\Gamma_+ \|u\|_\Omega^2 + \frac{1}{\Gamma_+} \|f\|_\Omega^2\right] \tag{39}$$

for any $\Omega' \subset \Omega$ with $\mathrm{dist}(\Omega', \partial\Omega) \ge \delta$.

Proof. We consider first the case when Ω and Ω' are open cubes, say $\Omega = \Lambda_L(x_0)$, $\Omega' = \Lambda_{L-2\delta}(x_0)$, for some $x_0 \in \mathbb{R}^d$, $L > 2\delta$. We fix $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \phi(x) \le 1$, $\phi(x) \equiv 1$ in Ω', $\phi(x) \equiv 0$ in $\mathbb{R}^d \backslash \Lambda_{L-\frac{\delta}{2}}(x_0)$, and $|(\nabla\phi)(x)| \le \frac{2\sqrt{d}}{\delta}$. (Such a function always exists.) We set $D\phi = \{D_{\alpha,\beta}\phi\}_{\alpha,\beta=1,\ldots,\nu} = \{a_{\alpha,\beta} \cdot \nabla\phi\}_{\alpha,\beta=1,\ldots,\nu}$.

Since $\phi^2 u \in \mathcal{D}(D_\Omega)$ with compact support, it follows from (38) that

$$\langle D\phi^2 u, \Gamma D u\rangle_\Omega = \langle \phi^2 u, f\rangle_\Omega, \tag{40}$$

so we have

$$\begin{aligned} 0 \le \langle Du, \phi^2 \Gamma D u\rangle_\Omega &= \langle \phi^2 u, f\rangle_\Omega - 2\langle (D\phi)u, \phi\Gamma D u\rangle_\Omega && (41)\\ &\le \|u\|_\Omega \|f\|_\Omega + 2\langle (D\phi)u, \Gamma(D\phi)u\rangle_\Omega^{\frac12} \langle Du, \phi^2\Gamma Du\rangle_\Omega^{\frac12} && (42)\\ &\le \left(\frac{\Gamma_+}{2}\|u\|_\Omega^2 + \frac{1}{2\Gamma_+}\|f\|_\Omega^2\right) \\ &\quad + \left(2\Gamma_+ C_\delta \|u\|_\Omega^2 + \frac12 \langle Du, \phi^2\Gamma Du\rangle_\Omega\right), && (43) \end{aligned}$$

where we used the elementary inequality $ab \le r^2a^2 + s^2b^2$, for any $a, b \ge 0$, $r, s > 0$ with $2rs = 1$, and $C_\delta = C(d, \nu, \{a_{\alpha,\beta}\}_{\alpha,\beta=1,\ldots,\nu}, \delta) < \infty$ is a constant depending only on the indicated parameters.

Thus,

$$\langle Du, \Gamma Du\rangle_{\Omega'} \le \langle Du, \phi^2\Gamma Du\rangle_\Omega \le \frac{1}{\Gamma_+}\|f\|_\Omega^2 + (1 + 4C_\delta)\Gamma_+\|u\|_\Omega^2, \tag{44}$$

which implies (39) when Ω is an open cube.

We now consider the general case: let Ω and Ω' be as in the theorem, with $\operatorname{dist}(\Omega', \partial\Omega) \ge \delta$ (we use the norm $|\ |_\infty$), and let

$$\Omega'_\delta = \{x \in \frac{\delta}{2}\mathbb{Z}^d;\ \ \Lambda_{\frac{\delta}{2}}(x) \cap \Omega' \ne \emptyset\}. \tag{45}$$

Using (44), we get

$$\begin{aligned} \langle Du, \Gamma Du\rangle_{\Omega'} &\le \sum_{x\in\Omega'_\delta} \langle Du, \Gamma Du\rangle_{\Lambda_{\frac{\delta}{2}}(x)} && (46)\\ &\le \sum_{x\in\Omega'_\delta}\left(\frac{1}{\Gamma_+}\|f\|_{\Lambda_\delta(x)}^2 + (1 + 4C_{\frac{\delta}{4}})\Gamma_+\|u\|_{\Lambda_\delta(x)}^2\right) && (47)\\ &\le (2d+1)\left(\frac{1}{\Gamma_+}\|f\|_\Omega^2 + (1 + 4C_{\frac{\delta}{4}})\Gamma_+\|u\|_\Omega^2\right), && (48) \end{aligned}$$

from which (39) follows. □

Theorem 11 has the following immediate corollaries for Maxwell operators. In this case $\nu = 3$, $D = \nabla^\times$ (i.e., $D\Psi = \nabla \times \Psi$), $D^* = D$, $D^*D = \Xi$, and $\Gamma = \frac{1}{\varepsilon}I_3$. We write $\nabla^\times|_\Omega$ for $(\nabla^\times)_\Omega$. If M on $L^2(\mathbb{R}^3; \mathbb{C}^3)$ be given by (17) with (14), we have $M = D^*\Gamma D$.

Corollary 12. *Let the operator M on $L^2(\mathbb{R}^3;\mathbb{C}^3)$ be given by (17) with (14). For any $\delta > 0$ there exists $\Theta_\delta < \infty$, depending only on δ, such that if $\Psi \in \mathcal{D}(\nabla^\times|_\Omega)$ is a weak solution for the equation $M\Psi = F$ in an open subset Ω of $\mathbb{R}^3$, with $F \in L^2(\Omega;\mathbb{C}^3)$, we have*

$$\|\nabla \times \Psi\|_{\Omega'} \leq \Theta_\delta \sqrt{\varepsilon_+}\left[\frac{1}{\sqrt{\varepsilon_-}}\|\Psi\|_\Omega + \sqrt{\varepsilon_-}\|F\|_\Omega\right] \tag{49}$$

for any $\Omega' \subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) \geq \delta$.

Corollary 13. *Let the operator M be given by (17) with (14). Let $\Psi \in L^2(\mathbb{R}^3;\mathbb{C}^3)$ be such that $\nabla \times \Psi$ is locally in L^2, i.e., $\Psi|_\Omega \in \mathcal{D}(\nabla^\times|_\Omega)$ for any bounded open $\Omega \subset \mathbb{R}^3$. Then, if Ψ is a weak solution for the equation $\mathbf{M}\Psi = F$ in $\mathbb{R}^3$, with $F \in L^2(\mathbb{R}^3;\mathbb{C}^3)$, we have*

$$\|\nabla \times \Psi\| \leq \Theta_\infty \sqrt{\varepsilon_+}\left[\frac{1}{\sqrt{\varepsilon_-}}\|\Psi\| + \sqrt{\varepsilon_-}\|F\|\right], \tag{50}$$

with $\Theta_\infty = \inf_{\delta>0}\Theta_\delta$.

Corollary 12 gives exponential decay for the curl of an exponentially decaying eigenfunction of a Maxwell operator.

Corollary 14. *Let $\mathbf{M}$ be an operator of the form (15) satisfying the bounds (14), and let Ψ be an eigenfunction for $\mathbf{M}$. Suppose Ψ has exponentially decaying local L^2-norms, i.e., $\|\chi_{x,\ell}\Psi\|_2$ decays exponentially as $\|x\| \to \infty$ for some $\ell > 0$. Then $\nabla \times \Psi$ also has exponentially decaying local L^2-norms.*

2.2. A Combes-Thomas argument. Let the operator M be given by (17). If $z \notin \sigma(M)$, we write $R(z) = (M - z)^{-1}$.

Lemma 15. *Let the operator M be given by (17) with (14). Then for any $z \notin \sigma(M)$, $n \in \mathbb{N}$ and $\ell > 0$ we have*

$$\|\chi_{x,\ell}R(z)^n\chi_{y,\ell}\| \;\leq \left(\frac{9}{\eta}\right)^n e^{(\sqrt{3}\ell/4)}e^{-m_z|x-y|} \quad \textit{for all } \; x,y \in \mathbb{R}^3, \tag{51}$$

with

$$m_z = \frac{\eta}{4\left[\varepsilon_-^{-1} + |z| + \eta\right]}, \tag{52}$$

where $\eta = \operatorname{dist}(z,\sigma(M))$.

Proof. The lemma is proved in the same way as [FK3, Lemma 12], with the obvious modifications to take into account that in this lemma we have *curls* instead of *gradients*. □

The next lemma gives an exponential estimate for the curl of the resolvent.

Lemma 16. *Let the operator M be given by (17) with (14), and let $z \notin \sigma(M)$ with $\eta,\ m_z$ as in Lemma 1. Then $\nabla^\times R(z)$ is a bounded operator on $L^2(\mathbb{R}^3,\mathbb{C}^3)$ with*

$$\left\|\nabla^\times R(z)\right\| \leq \Theta_1 \sqrt{\varepsilon_+}\left(\sqrt{\varepsilon_-} + \frac{1}{\sqrt{\varepsilon_-}}\right)\left(\frac{(1+|z|)}{\eta} + 1\right), \tag{53}$$

where Θ_1 is given in (49). Furthermore, for each $\ell > 0$ we have

$$\left\|\chi_{x,\ell}\nabla^{\times}R(z)\chi_{y,\ell}\right\| \leq \Theta_1\sqrt{\varepsilon_+}\left(\sqrt{\varepsilon_-}+\frac{1}{\sqrt{\varepsilon_-}}\right)(1+|z|)\frac{9}{\eta}e^{(3\sqrt{3}\ell/4)}e^{-m_z|x-y|} \quad (54)$$

for all $x,\ y \in \mathbb{R}^3$ with $|x-y| \geq 2\ell$.

Proof. This lemma is proven in the same way as [FK3, Lemma 13], using Corollaries 12 and 13, and Lemma 15. □

2.3. Generalized eigenfunctions. Let M be an operator of the form (17) satisfying the bounds (14). Given $z \in \mathbb{C}$, a measurable function $\Psi : \mathbb{R}^3 \to \mathbb{C}^3$ will be called a *generalized eigenfunction* for z if both Ψ and $\nabla \times \Psi$ are locally in L^2, i.e., $\Psi|_{\Omega} \in \mathcal{D}(\nabla^{\times}|_{\Omega})$ for all open bounded subsets Ω of $\mathbb{R}^3$, and Ψ is a weak solution for the equation $M\Psi = z\Psi$ on $\mathbb{R}^3$, i.e.,

$$\langle \nabla \times \Phi, \frac{1}{\varepsilon}\nabla \times \Psi\rangle = z\langle \Phi, \Psi\rangle \ \text{ for all } \ \Phi \in C_0^{\infty}(\mathbb{R}^3; \mathbb{C}^3). \quad (55)$$

Theorem 17. *Let M be an operator of the form (17) satisfying the bounds (14), $\rho(d\lambda)$ its spectral measure. Let $w(x) = \left(|x|^p + 1\right)^{-1}$ with $p > 3$. Then, for $\rho(d\lambda)$-almost all $\lambda > 0$, M has a generalized eigenfunction Ψ_λ satisfying*

$$\int_{\mathbb{R}^3} |\Psi_\lambda(x)|^2 w(x)\, dx < \infty, \quad (56)$$

so for any $\ell \in \mathbb{N}$ we have

$$\|\chi_{x,\ell}\Psi_\lambda\| \leq C_\ell\left(|x|^p + 1\right) \ \textit{for all } x \in \ell\mathbb{Z}^3, \quad (57)$$

for some constant $C_\ell < \infty$ depending only on ℓ, $\varepsilon_\pm$ and the LHS of (56).

Proof. Let

$$F(t) = \begin{cases} (t+1)^{-1}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases} \quad (58)$$

F is a bounded measurable function on the real line, continuous on $(0,\infty)$, such that

$$F(M) = (\mathbf{M} + I_{\mathbb{S}})^{-1} \oplus 0_{\mathbb{G}} \quad (59)$$

with respect to Weyl's decomposition (19).

The operator $F(M)W^{\frac{1}{2}}$ is Hilbert-Schmidt by Theorem 18 below, W being the operator given by multiplication by the function $w(x)$. The existence of generalized eigenfunctions satisfying (56), for $\rho(d\lambda)$-almost all $\lambda > 0$, now follows from [B, Subsects. V.4.1–V.4.2].

The estimate (57) is an immediate consequence of (56). □

2.4. Estimates on traces.

Theorem 18. *Let M be an operator of the form (17) with (14), and let V denote the bounded operator given by multiplication by the bounded measurable function $v(x)$, with $v(x) \geq 0$ and*

$$\sum_{x \in \mathbb{Z}^3} \|\chi_{x,1} v^2\|_\infty < \infty. \tag{60}$$

Then the operator

$$P_{\mathbb{S}}(M + I)^{-1} V = \left[(\mathbf{M} + I_{\mathbb{S}})^{-1} \oplus 0_{\mathbb{G}}\right] V \tag{61}$$

is Hilbert-Schmidt.

Theorem 18 was used in the proof of Theorem 17 with $v(x) = [w(x)]^{\frac{1}{2}} = \left(|x|^p + 1\right)^{-\frac{1}{2}}$.

To prove the theorem we will introduce a modified Maxwell operator $\widetilde{M}$ which is elliptic. Formally,

$$\widetilde{M} = \widetilde{M}(\varepsilon) = M + Y, \tag{62}$$

with $Y = -\nabla \frac{1}{\varepsilon} \nabla \cdot$, i.e., $Y\Psi = -\nabla \left\{\frac{1}{\varepsilon}[\nabla \cdot \Psi]\right\}$. $\widetilde{M}$ is rigorously defined as the nonnegative self-adjoint operator on $L^2(\mathbb{R}^3; \mathbb{C}^3)$ given by the closure of the nonnegative quadratic form

$$\widetilde{\mathcal{M}}[\Psi] = \mathcal{M}[\Psi] + \int_{\mathbb{R}^3} \frac{1}{\varepsilon(x)} \left|[\nabla \cdot \Psi](x)\right|^2 \, dx, \quad \Psi \in C_0^1(\mathbb{R}^3; \mathbb{C}^3). \tag{63}$$

The operator $\widetilde{M}$ is diagonal with respect to Weyl's decomposition (19), with $\widetilde{M} = \mathbf{M} \oplus \mathbf{Y}$ for the appropriate operator $\mathbf{Y}$ on $\mathbb{G}$.

If $\varepsilon(x) \equiv 1$, we have

$$\widetilde{\Xi} \equiv \widetilde{M}(1) = -\Delta \otimes I_3, \tag{64}$$

where Δ is the Laplacian in $L^2(\mathbb{R}^3)$ and I_3 is the identity operator on $\mathbb{C}^3$.

Since

$$(\mathbf{M} + I_{\mathbb{S}})^{-2} \oplus 0_{\mathbb{G}} \leq (\mathbf{M} + I_{\mathbb{S}})^{-2} \oplus (\mathbf{Y} + I_{\mathbb{G}})^{-2} = \left(\widetilde{M} + I\right)^{-2}, \tag{65}$$

Theorem 18 is an immediate consequence of the following theorem.

Theorem 19. *Let $\widetilde{M}$ be as in (62) with (14), and let $v(x)$ and V be as in Theorem 18. Then the operator $\left(\widetilde{M} + I\right)^{-1} V$ is Hilbert-Schmidt.*

Proof. We set $\chi_x = \chi_{x,1}$ for $x \in \mathbb{R}^3$ and

$$\widetilde{R} = \left(\widetilde{M} + 1\right)^{-1}, \quad \widetilde{S}(\mu) = \left(\widetilde{\Xi} + \mu\right)^{-1} \quad \text{for} \quad \mu > 0; \tag{66}$$

$$\widetilde{R}_{x,y} = \chi_x \widetilde{R} \chi_y, \quad \widetilde{S}(\mu)_{x,y} = \chi_x \widetilde{S}(\mu) \chi_y \quad \text{for} \quad x, y \in \mathbb{R}^3. \tag{67}$$

It follows from (14) and (75) that

$$\varepsilon_- \widetilde{S}(\varepsilon_-) \leq \widetilde{R} \leq \varepsilon_+ \widetilde{S}(\varepsilon_+). \tag{68}$$

We let

$$\widehat{S} = \varepsilon_+ \widetilde{S}(\varepsilon_+), \quad \widehat{S}_{x,y} = \chi_x \widehat{S}(\mu) \chi_y. \tag{69}$$

We also set $\chi_{x,y} = \max\{\chi_x, \chi_y\}$ for $x, y \in \mathbb{R}^3$, notice $\chi_{x,y}^2 = \chi_{x,y}$ and $\chi_{x,x} = \chi_x$.

Lemma 20. *Let $p > \frac{3}{2}$ and $\mu > 0$. Then there exists a constant $c_1 = c_1(p,\mu) < \infty$, depending only on the indicated parameters, such that*

$$\operatorname{Tr}\left\{\chi_{x,y}\left[\widetilde{S}(\mu)\right]^p \chi_{x,y}\right\} \le c_1 \tag{70}$$

for all $x, y \in \mathbb{R}^3$.

Proof. It follows from (64) that it suffices to show that

$$\operatorname{Tr}\left\{\chi_{x,y}\,(-\Delta+\mu)^{-p}\,\chi_{x,y}\right\} \le c \tag{71}$$

for all $x, y \in \mathbb{R}^3$, the trace now being calculated in $L^2(\mathbb{R}^3)$. But this is a consequence of the fact that the operator $(-\Delta+\mu)^{-p}$ has a bounded kernel; it is taken into multiplication by an integrable function by the Fourier transform. □

We recall some general results. Given a compact operator A on a Hilbert space, we set $s_j(A) = \lambda_j(|A|)$, where $\lambda_1\left(|A|\right) \ge \lambda_2\left(|A|\right) \ge \ldots$ are the strictly positive eigenvalues of $|A|$, repeated according to their multiplicity. For such A we have (e.g., [GK]):

$$\|A\|_p^p \equiv \operatorname{Tr}\left(|A|^p\right) = \sum_j [s_j(A)]^p, \quad 1 \le p < \infty; \tag{72}$$

$$s_j(A) = s_j\left(A^*\right) \quad \text{for any } j, \quad \text{so} \quad \|A\|_p = \|A^*\|_p; \tag{73}$$

$$s_j(BA), s_j(AB) \le \|B\|\, s_j(A) \quad \text{for any bounded operator } B. \tag{74}$$

If A and B are self-adjoint operators and $A \ge B \ge 0$, we have

$$\operatorname{Tr} A^2 \ge \operatorname{Tr} B^2; \quad A^{-1} \le B^{-1}; \quad A^\beta \ge B^\beta,\ 0 < \beta \le 1. \tag{75}$$

We will also need the following general statement.

Lemma 21. *Let A be a nonnegative bounded operator and P an orthogonal projection on a Hilbert space $\mathcal{H}$. For any $\gamma \ge 1$ we have*

$$\operatorname{Tr}[PAP]^\gamma \le \operatorname{Tr} PA^\gamma P. \tag{76}$$

Proof. Let B be a nonegative compact operator on $\mathcal{H}$. By the mini-max principle, we get

$$\lambda_j(B) = \max_{\{F\subset\mathcal{H};\,\dim F = j\}}\ \min_{\{\varphi\in F;\,\|\varphi\|=1\}} \langle\varphi, B\varphi\rangle\,. \tag{77}$$

If $\gamma \ge 1$, it follows from Jensen's inequality that for any $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ we have

$$\langle\varphi, B\varphi\rangle^\gamma \le \langle\varphi, B^\gamma\varphi\rangle\,. \tag{78}$$

Without loss of generality we can assume $\operatorname{Tr} PA^\gamma P < \infty$. In this case we claim that

$$[\lambda_j(PAP)]^\gamma \le \lambda_j(PA^\gamma P) \quad \text{for any } j, \tag{79}$$

so (76) follows. Indeed, using (78) and (77) we obtain, with $\mathcal{F} = P\mathcal{H}$,

$$
\begin{aligned}
[\lambda_j(PAP)]^\gamma &= \left[\max_{\{F\subset\mathcal{F};\,\dim F=j\}}\ \min_{\{\varphi\in F;\,\|\varphi\|=1\}} \langle\varphi, B\varphi\rangle\right]^\gamma \\
&= \max_{\{F\subset\mathcal{F};\,\dim F=j\}}\ \min_{\{\varphi\in F;\,\|\varphi\|=1\}} \langle\varphi, B\varphi\rangle^\gamma \\
&\le \max_{\{F\subset\mathcal{F};\,\dim F=j\}}\ \min_{\{\varphi\in F;\,\|\varphi\|=1\}} \langle\varphi, B^\gamma\varphi\rangle \\
&= \lambda_j(PA^\gamma P).
\end{aligned}
$$

□

Lemma 22. *There exists a constant $c_2 = c_2(\varepsilon_+) < \infty$, depending only on ε_+, such that*

$$\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^2 = \operatorname{Tr}\widetilde{R}^*_{x,y}\widetilde{R}_{x,y} \le c_2 \quad \textit{for all } x, y \in \mathbb{R}^3. \tag{80}$$

In particular, the operators $\widetilde{R}_{x,y}$ are compact.

Proof. We have

$$
\begin{aligned}
\operatorname{Tr}\widetilde{R}^*_{x,y}\widetilde{R}_{x,y} &= \operatorname{Tr}\chi_y\widetilde{R}\chi_x\widetilde{R}\chi_y \le \operatorname{Tr}\chi_y\widetilde{R}\chi_{x,y}\widetilde{R}\chi_y \\
&= \operatorname{Tr}\chi_{x,y}\widetilde{R}\chi_y\widetilde{R}\chi_{x,y} \le \operatorname{Tr}\chi_{x,y}\widetilde{R}\chi_{x,y}\widetilde{R}\chi_{x,y} = \operatorname{Tr}\left(\chi_{x,y}\widetilde{R}\chi_{x,y}\right)^2.
\end{aligned} \tag{81}
$$

On the other hand using (75), (68), (69) and (70) we obtain

$$
\begin{aligned}
\operatorname{Tr}\left(\chi_{x,y}\widetilde{R}\chi_{x,y}\right)^2 &\le \operatorname{Tr}\left(\chi_{x,y}\widehat{S}\chi_{x,y}\right)^2 = \operatorname{Tr}\chi_{x,y}\widehat{S}\chi_{x,y}\widehat{S}\chi_{x,y} \\
&\le \operatorname{Tr}\chi_{x,y}\widehat{S}^2\chi_{x,y} \le \varepsilon_+^2 c_1(2, \varepsilon_+).
\end{aligned} \tag{82}
$$

The inequalities (81) and (82) imply (80). □

Lemma 23. *There exists a constant $c_3 = c_3\left(\varepsilon_\pm\right) < \infty$, depending only on $\varepsilon_\pm$, such that*

$$\operatorname{Tr}\chi_x\widetilde{R}^2\chi_x \le c_3 \quad \textit{for all } x \in \mathbb{R}^3. \tag{83}$$

Proof. We have

$$\operatorname{Tr}\chi_x\widetilde{R}^2\chi_x = \sum_{y\in\mathbb{Z}^3}\operatorname{Tr}\chi_x\widetilde{R}\chi_y\widetilde{R}\chi_x = \sum_{y\in\mathbb{Z}^3}\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^2. \tag{84}$$

In addition, if $0 \le \alpha < 1$, we also have

$$\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^2 = \operatorname{Tr}\left\{\left|\widetilde{R}_{x,y}\right|^{1-\frac{\alpha}{2}}\left|\widetilde{R}_{x,y}\right|^{\alpha}\left|\widetilde{R}_{x,y}\right|^{1-\frac{\alpha}{2}}\right\} \le \left\|\widetilde{R}_{x,y}\right\|^{\alpha}\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^{2-\alpha}, \tag{85}$$

so

$$
\begin{aligned}
\operatorname{Tr}\chi_x\widetilde{R}^2\chi_x &\le \sum_{y\in\mathbb{Z}^3}\left\|\widetilde{R}_{x,y}\right\|^{\alpha}\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^{2-\alpha} \\
&\le \left[\sup_{y\in\mathbb{Z}^3}\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^{2-\alpha}\right]\sum_{y\in\mathbb{Z}^3}\left\|\widetilde{R}_{x,y}\right\|^{\alpha}.
\end{aligned} \tag{86}
$$

From Lemma 15, which holds exactly as stated with $\widetilde{M}$ substituted for M, we get that

$$\sum_{y\in\mathbb{Z}^3} \left\|\widetilde{R}_{x,y}\right\|^{\alpha} \leq b(\varepsilon_-,\alpha) \tag{87}$$

for some constant $b(\varepsilon_-,\alpha) < \infty$, which depends only on ε_- and α. To estimate the other term, notice that using (73), (74), (75), (68) and (69), we obtain

$$\begin{aligned}
&\left[s_j\left(\widetilde{R}_{x,y}\right)\right]^2 = s_j\left(\left|\widetilde{R}_{x,y}\right|^2\right)\\
&= s_j\left(\chi_y\widetilde{R}\chi_x\widetilde{R}\chi_y\right) \leq s_j\left(\chi_y\widetilde{R}\chi_{x,y}\widetilde{R}\chi_y\right)\\
&= s_j\left(\chi_y\chi_{x,y}\widetilde{R}\chi_{x,y}\widetilde{R}\chi_{x,y}\chi_y\right) \leq s_j\left(\chi_{x,y}\widetilde{R}\chi_{x,y}\widetilde{R}\chi_{x,y}\right) = s_j\left(\left[\chi_{x,y}\widetilde{R}\chi_{x,y}\right]^2\right)\\
&= \left[s_j\left(\chi_{x,y}\widetilde{R}\chi_{x,y}\right)\right]^2 \leq \left[s_j\left(\chi_{x,y}\widehat{S}\chi_{x,y}\right)\right]^2 = s_j\left(\left[\chi_{x,y}\widehat{S}\chi_{x,y}\right]^2\right)\\
&= s_j\left(\chi_{x,y}\widehat{S}\chi_{x,y}\widehat{S}\chi_{x,y}\right) = \left[s_j\left(\chi_{x,y}\widehat{S}\chi_{x,y}\right)\right]^2 .
\end{aligned} \tag{88}$$

Taking $\alpha \in \left(0,1/2\right)$ so $2-\alpha > \frac{3}{2}$, we use (76) and (70) to get

$$\begin{aligned}
\operatorname{Tr}\left|\widetilde{R}_{x,y}\right|^{2-\alpha} &= \sum_j \left[s_j\left(\widetilde{R}_{x,y}\right)\right]^{2-\alpha} \leq \sum_j \left[s_j\left(\chi_{x,y}\widehat{S}\chi_{x,y}\right)\right]^{2-\alpha}\\
&= \operatorname{Tr}\left[\chi_{x,y}\widehat{S}\chi_{x,y}\right]^{2-\alpha} \leq \operatorname{Tr}\chi_{x,y}\widehat{S}^{2-\alpha}\chi_{x,y}\\
&\leq c_1\left(2-\alpha,\varepsilon_+\right).
\end{aligned} \tag{89}$$

The lemma is proved, since (83) follows from (86), (87) and (89). $\square$

We can now finish the proof of Theorem 19. Using (83) and (60), we get

$$\begin{aligned}
\operatorname{Tr} V\widetilde{R}^2 V = \operatorname{Tr}\widetilde{R}V^2\widetilde{R} &= \sum_{x\in\mathbb{Z}^3}\operatorname{Tr}\widetilde{R}\chi_x V^2\widetilde{R} \leq \sum_{x\in\mathbb{Z}^3}\|\chi_x v^2\|_\infty \operatorname{Tr}\widetilde{R}\chi_x\widetilde{R}\\
&= \sum_{x\in\mathbb{Z}^3}\|\chi_x v^2\|_\infty \operatorname{Tr}\chi_x\widetilde{R}^2\chi_x \leq c_3\sum_{x\in\mathbb{Z}^3}\|\chi_x v^2\|_\infty < \infty,
\end{aligned} \tag{90}$$

so $\widetilde{R}V$ is a Hilbert-Schmidt operator. $\square$

3. Periodic Maxwell Operators and Periodic Boundary Condition

The (non-random) spectrum of a random Maxwell operator can be represented as the union of the spectra of relevant periodic Maxwell operators, which in turn are given as the union of the spectra of finite volume Maxwell operators with periodic boundary condition. This is analogous to the situation for random Schrödinger operators [KM2] and random acoustic operators [FK3].

In this section we study Maxwell operators in periodic media. We say that the operators **M**, M, given by (15), (17) with (14), are q-periodic for some $q > 0$, if

$\varepsilon(x)$ is a q-periodic function. In this section we work with a given period $q > 0$ and q-periodic operators **M** and M.

3.1. Periodic boundary condition. We start by defining the restriction of such M to a cube with periodic boundary condition. Given a cube $\Lambda = \Lambda_\ell(x)$, where $x \in \mathbb{R}^3$ and $\ell > 0$, we will denote by $\overset{\circ}{\Lambda}$ the torus we obtain by identifying the edges of the closed cube $\bar{\Lambda}$ in the usual way. We introduce the usual distance in the torus:

$$\overset{\circ}{d}(x,y) \equiv \min_{m \in \ell\mathbb{Z}^3} |x - y + m| \leq \frac{\sqrt{3}\ell}{2} \text{ for all } x, y \in \bar{\Lambda}. \tag{91}$$

We will identify functions on $\overset{\circ}{\Lambda}$ with their ℓ-periodic extensions to $\mathbb{R}^3$; for example, $C^1\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right)$ will be identified with the space of continuously differentiable, ℓ-periodic, $\mathbb{C}^3$-valued functions on $\mathbb{R}^3$. We define $W^{1,2}\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right)$ as the closure of $C^1\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right)$ in $W^{1,2}\left(\Lambda; \mathbb{C}^3\right)$.

We will always take $\ell \in q\mathbb{N}$ and define $\overset{\circ}{M}_\Lambda$, the restriction of M to Λ with periodic boundary condition, as the unique nonnegative self-adjoint operator on $L^2\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right) \cong L^2\left(\Lambda; \mathbb{C}^3\right)$, defined by the nonnegative densely defined closed quadratic form

$$\overset{\circ}{\mathcal{M}}_\Lambda(\Psi, \Phi) = \langle \nabla \times \Psi, \frac{1}{\varepsilon} \nabla \times \Phi \rangle, \quad \text{with} \quad \Psi, \Phi \in W^{1,2}\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right)_\Lambda, \tag{92}$$

the inner product being in $L^2\left(\Lambda; \mathbb{C}^3\right)$.

We also have a corresponding Weyl's decomposition in the torus: $L^2\left(\Lambda; \mathbb{C}^3\right) = \overset{\circ}{\mathbb{S}}_\Lambda \oplus \overset{\circ}{\mathbb{G}}_\Lambda$, where

$$\overset{\circ}{\mathbb{S}}_\Lambda = \overline{\left\{\Psi \in L^2\left(\Lambda; \mathbb{C}^3\right); \quad \Psi \in C^1\left(\overset{\circ}{\Lambda}; \mathbb{C}^3\right) \quad \text{with} \quad \nabla \cdot \Psi = 0\right\}}, \tag{93}$$

$$\overset{\circ}{\mathbb{G}}_\Lambda = \overline{\left\{\Psi \in L^2\left(\Lambda; \mathbb{C}^3\right); \quad \Psi = \nabla\varphi \quad \text{with} \quad \varphi \in C^1\left(\overset{\circ}{\Lambda}\right)\right\}}. \tag{94}$$

The spaces $\overset{\circ}{\mathbb{S}}_\Lambda$ and $\overset{\circ}{\mathbb{G}}_\Lambda$ are left invariant by $\overset{\circ}{M}_\Lambda$, with $\overset{\circ}{\mathbb{G}}_\Lambda \subset \mathcal{D}\left(\overset{\circ}{M}_\Lambda\right)$ and $\overset{\circ}{M}_\Lambda\Big|_{\overset{\circ}{\mathbb{G}}_\Lambda} = 0$. We define $\overset{\circ}{\mathbf{M}}_\Lambda$ as the restriction of $\overset{\circ}{M}_\Lambda$ to $\overset{\circ}{\mathbb{S}}_\Lambda$, i.e., $\mathcal{D}\left(\overset{\circ}{\mathbf{M}}_\Lambda\right) = \mathcal{D}\left(\overset{\circ}{M}_\Lambda\right) \cap \overset{\circ}{\mathbb{S}}_\Lambda$ and $\overset{\circ}{\mathbf{M}}_\Lambda = \overset{\circ}{M}_\Lambda\Big|_{\mathcal{D}\left(\overset{\circ}{M}_\Lambda\right) \cap \overset{\circ}{\mathbb{S}}_\Lambda}$. Thus $\overset{\circ}{\mathbf{M}}_\Lambda = P_{\overset{\circ}{\mathbb{S}}_\Lambda} \overset{\circ}{M}_\Lambda I_{\overset{\circ}{\mathbb{S}}_\Lambda} = \overset{\circ}{M}_\Lambda I_{\overset{\circ}{\mathbb{S}}_\Lambda}$, with $P_{\overset{\circ}{\mathbb{S}}_\Lambda}$ the orthogonal projection onto $\overset{\circ}{\mathbb{S}}_\Lambda$ and $I_{\overset{\circ}{\mathbb{S}}_\Lambda} : \overset{\circ}{\mathbb{S}}_\Lambda \to L^2\left(\Lambda; \mathbb{C}^3\right)$ the restriction of the identity map. Notice that $\overset{\circ}{M}_\Lambda = \overset{\circ}{\mathbf{M}}_\Lambda \oplus 0_{\overset{\circ}{\mathbb{G}}_\Lambda}$, and 0 is easily seen to be an eigenvalue of $\overset{\circ}{\mathbf{M}}_\Lambda$ with multiplicity three, so

$$\sigma\left(\overset{\circ}{\mathbf{M}}_\Lambda\right) = \sigma\left(\overset{\circ}{M}_\Lambda\right). \tag{95}$$

If $\varepsilon(x) \equiv 1$ we write $\overset{\circ}{\Xi}_\Lambda$, $\overset{\circ}{\Xi}_\Lambda$ for $\overset{\circ}{M}_\Lambda$, $\overset{\circ}{\mathbf{M}}_\Lambda$, respectively. Since $\overset{\circ}{\Xi}_\Lambda$ has compact resolvent (its eigenvalues and eigenfunctions can be explicitly computed), and $\overset{\circ}{\mathbf{M}}_\Lambda \geq \frac{1}{\varepsilon_+}\overset{\circ}{\Xi}_\Lambda$ by (14), we can conclude that $\overset{\circ}{\mathbf{M}}_\Lambda$ has compact resolvent.

3.2. A Combes-Thomas argument for the torus. If $z \notin \sigma(\overset{\circ}{M}_\Lambda)$, we write $\overset{\circ}{R}_\Lambda(z) = (\overset{\circ}{M}_\Lambda - z)^{-1}$.

Lemma 24. *Let the operator M given by (17) with (14) be q-periodic, and let $\Lambda = \Lambda_\ell(x_0)$ for some $x_0 \in \mathbb{R}^3$ and $\ell \in q\mathbb{N}$, $\ell > 2r + 8$, where $r > 0$. Then for any $z \notin \sigma(\overset{\circ}{M}_\Lambda)$ and $n \in \mathbb{N}$ we have*

$$\|\chi_{x,r}\overset{\circ}{R}_\Lambda(z)^n \chi_{y,r}\| \leq \left(\frac{9}{\eta}\right)^n e^{\frac{\sqrt{3}r\overset{\circ}{m}_{z,r,\ell}}{2}} e^{-\overset{\circ}{m}_{z,r,\ell}\overset{\circ}{d}(x,y)} \quad \textit{for all} \ \ x, y \in \overset{\circ}{\Lambda}, \tag{96}$$

with

$$\overset{\circ}{m}_{z,r,\ell} = \frac{\eta}{4\left(\frac{2\sqrt{3}}{1-\frac{2r+8}{\ell}} + 1\right)\left[\varepsilon_-^{-1} + |z| + \eta\right]}, \tag{97}$$

where $\eta = \operatorname{dist}(z, \sigma(\overset{\circ}{M}_\Lambda))$.

Proof. The lemma is proved in the same way as [FK3, Lemma 18], with the obvious modifications to take into account that in this lemma we have *curls* instead of *gradients*. □

3.3. Floquet theory and the spectrum of periodic operators. If $k, n \in \mathbb{N}$, we say that $k \preceq n$ if $n \in k\mathbb{N}$ and that $k \prec n$ if $k \preceq n$ and $k \neq n$. The main result of this section is the following theorem.

Theorem 25. *Suppose the operator $\mathbf{M}$ given by (15) with (14) is q-periodic. Let $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Then*

$$\sigma\left(\overset{\circ}{\mathbf{M}}_{\Lambda_{\ell_n}(0)}\right) \subset \sigma\left(\overset{\circ}{\mathbf{M}}_{\Lambda_{\ell_{n+1}}(0)}\right) \subset \sigma(\mathbf{M}) \ \ \textit{for all} \ \ n = 0, 1, 2, \ldots, \tag{98}$$

and

$$\sigma(\mathbf{M}) = \overline{\bigcup_{n \geq 1} \sigma\left(\overset{\circ}{\mathbf{M}}_{\Lambda_{\ell_n}(0)}\right)}. \tag{99}$$

Related results for periodic Schrödinger operators can be found in [Ea], where Floquet theory is used. For the nonsmooth coefficients we are interested in some aspects of the Floquet theory have to be revised. Periodic acoustic operators are treated in [FK3, Theorem 14], with a proof that does not use Floquet theory. In this subsection we will develop an appropriate Floquet theory for our Maxwell operators, and use it to prove Theorem 25. We refer to [RS4, Sect. XIII.6] for the definitions and notations of direct integrals of Hilbert spaces.

Let $Q = \breve{\Lambda}_q(0)$ be the basic period cell, $\tilde{Q} = \breve{\Lambda}_{\frac{2\pi}{q}}(0)$ the dual basic cell. We define the Floquet transform

$$\mathcal{F}: \quad L^2\left(\mathbb{R}^3; \mathbb{C}^3\right) \to \int_{\tilde{Q}}^{\oplus} L^2\left(Q; \mathbb{C}^3\right) \, dk \equiv L^2\left(\tilde{Q}; L^2\left(Q; \mathbb{C}^3\right)\right) \tag{100}$$

by

$$(\mathcal{F}\Psi)(k, x) = \left(\frac{q}{2\pi}\right)^{\frac{3}{2}} \sum_{m \in q\mathbb{Z}^3} \mathrm{e}^{ik\cdot(x-m)} \Psi(x - m), \quad x \in Q, \; k \in \tilde{Q}, \tag{101}$$

if Ψ has compact support; it extends by continuity to a unitary operator.

The q-periodic operator M is decomposable in this direct integral representation, more precisely,

$$\mathcal{F} M \mathcal{F}^* = \int_{\tilde{Q}}^{\oplus} \overset{\circ}{M}_Q(k) \, dk, \tag{102}$$

where for each $k \in \mathbb{R}^3$ we define $\overset{\circ}{M}_Q(k)$ to be the operator $(\nabla - ik)^{\times} \frac{1}{\varepsilon} (\nabla - ik)^{\times}$ on $L^2\left(Q; \mathbb{C}^3\right)$ with periodic boundary condition; $\overset{\circ}{M}_Q(k)$ is rigorously defined as a self-adjoint operator by the appropriate quadratic form $\overset{\circ}{\mathcal{M}}_Q(k)$ as in (92). As before $(\nabla - ik)^{\times}$ denotes the operator $(\nabla - ik)^{\times}\Phi = (\nabla - ik) \times \Phi$. We also have Weyl's decompositions for each $k \in \mathbb{R}^3$: $L^2\left(Q; \mathbb{C}^3\right) = \overset{\circ}{\mathbb{S}}_Q(k) \oplus \overset{\circ}{\mathbb{G}}_Q(k)$, where

$$\overset{\circ}{\mathbb{S}}_Q(k) = \overline{\left\{ \Psi \in L^2\left(Q; \mathbb{C}^3\right); \; \Psi \in C^1\left(\overset{\circ}{Q}; \, \mathbb{C}^3\right) \text{ with } (\nabla - ik) \cdot \Psi = 0 \right\}}, \tag{103}$$

$$\overset{\circ}{\mathbb{G}}_Q(k) = \overline{\left\{ \Psi \in L^2\left(Q; \mathbb{C}^3\right); \; \Psi = (\nabla - ik)\varphi \text{ with } \varphi \in C^1\left(\overset{\circ}{Q}\right) \right\}}. \tag{104}$$

The spaces $\overset{\circ}{\mathbb{S}}_Q(k)$ and $\overset{\circ}{\mathbb{G}}_Q(k)$ are left invariant by $\overset{\circ}{M}_Q(k)$, with $\overset{\circ}{\mathbb{G}}_Q(k) \subset \mathcal{D}\left(\overset{\circ}{M}_Q(k)\right)$ and $\overset{\circ}{M}_Q(k)\Big|_{\overset{\circ}{\mathbb{G}}_Q(k)} = 0$. We define $\overset{\circ}{\mathbf{M}}_Q(k)$ as the restriction of $\overset{\circ}{M}_Q(k)$ to $\overset{\circ}{\mathbb{S}}_Q(k)$, i.e., $\mathcal{D}\left(\overset{\circ}{\mathbf{M}}_Q(k)\right) = \mathcal{D}\left(\overset{\circ}{M}_Q\, k\right) \cap \overset{\circ}{\mathbb{S}}_Q(k)$ and $\overset{\circ}{\mathbf{M}}_Q(k) = \overset{\circ}{M}_Q(k)\Big|_{\mathcal{D}\left(\overset{\circ}{M}_Q(k)\right) \cap \overset{\circ}{\mathbb{S}}_Q(k)}$. Thus $\overset{\circ}{\mathbf{M}}_Q(k) = P_{\overset{\circ}{\mathbb{S}}_Q(k)} \overset{\circ}{M}_Q(k) I_{\overset{\circ}{\mathbb{S}}_Q(k)} = \overset{\circ}{M}_Q(k) I_{\overset{\circ}{\mathbb{S}}_Q(k)}$, with $P_{\overset{\circ}{\mathbb{S}}_Q(k)}$ the orthogonal projection onto $\overset{\circ}{\mathbb{S}}_Q(k)$ and $I_{\overset{\circ}{\mathbb{S}}_Q(k)} : \overset{\circ}{\mathbb{S}}_Q(k) \to L^2\left(Q; \, \mathbb{C}^3\right)$ the restriction of the identity map. Notice $\overset{\circ}{M}_Q(k) = \overset{\circ}{\mathbf{M}}_Q(k) \oplus 0_{\overset{\circ}{\mathbb{G}}_Q(k)}$, so $\sigma\left(\overset{\circ}{\mathbf{M}}_Q(k)\right) = \sigma\left(\overset{\circ}{M}_Q(k)\right)$. Each $\overset{\circ}{\mathbf{M}}_Q(k)$ has compact resolvent. We have

$$\mathcal{F}\mathbb{S} = \int_{\tilde{Q}}^{\oplus} \overset{\circ}{\mathbb{S}}_Q(k) \, dk, \quad \mathcal{F}\mathbf{M}\mathcal{F}^* = \int_{\tilde{Q}}^{\oplus} \overset{\circ}{\mathbf{M}}_Q(k) \, dk. \tag{105}$$

In addition, if for each $p \in \frac{2\pi}{q}\mathbb{Z}^3$ we let U_p denote the unitary operator on $L^2\left(Q; \mathbb{C}^3\right)$ given by multiplication by the function $\mathrm{e}^{-ip\cdot x}$, then for all $k \in \mathbb{R}^d$ and $p \in \frac{2\pi}{q}\mathbb{Z}^3$ we have

$$\overset{\circ}{M}_Q\,(k+p) = U_p^* \overset{\circ}{M}_Q\,(k) U_p, \tag{106}$$

and, since $U_p \overset{\circ}{\mathbb{S}}_Q(k+p) = \overset{\circ}{\mathbb{S}}_Q(k)$, we can also think of U_p as a unitary operator from $\overset{\circ}{\mathbb{S}}_Q(k+p)$ to $\overset{\circ}{\mathbb{S}}_Q(k)$, with

$$\overset{\circ}{\mathbf{M}}_Q\,(k+p) = U_p^* \overset{\circ}{\mathbf{M}}_Q\,(k) U_p. \tag{107}$$

Lemma 26. (i) *The mapping*

$$k \in \mathbb{R}^3 \quad \longmapsto \quad \overset{\circ}{R}_Q\,(k) \equiv \left(\overset{\circ}{M}_Q\,(k) + I\right)^{-1} \in \mathcal{L}\left(L^2\left(Q;\mathbb{C}^3\right)\right) \tag{108}$$

is operator norm continuous.
(ii) *We have*

$$\sigma(M) = \overline{\bigcup_{k\in\tilde{Q}} \sigma\left(\overset{\circ}{M}_Q\,(k)\right)} \quad \text{and} \quad \sigma(\mathbf{M}) = \overline{\bigcup_{k\in\tilde{Q}} \sigma\left(\overset{\circ}{\mathbf{M}}_Q\,(k)\right)}. \tag{109}$$

Proof. Let $k, h \in \mathbb{R}^3$, $\Psi \in L^2\left(Q;\mathbb{C}^3\right)$, we have

$$\begin{aligned} &\overset{\circ}{\mathcal{M}}_Q\,(k+h)[\Psi] - \overset{\circ}{\mathcal{M}}_Q\,(k)[\Psi] = \\ &\quad \langle h\times\Psi, \frac{1}{\varepsilon} h\times\Psi\rangle_Q + i\langle h\times\Psi, \frac{1}{\varepsilon}(\nabla - ik)\times\Psi\rangle_Q - i\langle(\nabla - ik)\times\Psi, \frac{1}{\varepsilon} h\times\Psi\rangle_Q. \end{aligned} \tag{110}$$

Using the Cauchy-Schwarz inequality and (14) we get (see [FK3, Proof of Lemma 12] for a similar argument)

$$\left|\overset{\circ}{\mathcal{M}}_Q\,(k+h)[\Psi] - \overset{\circ}{\mathcal{M}}_Q\,(k)[\Psi]\right| \le |h| \overset{\circ}{\mathcal{M}}_Q\,(k)[\Psi] + |h|\left(1+|h|\right)\frac{1}{\varepsilon_-}\|\Psi\|_Q^2. \tag{111}$$

If $|h| < 1$ we have

$$\begin{aligned} &\|(|h|(1+|h|)\varepsilon_-^{-1} + |h| \overset{\circ}{M}_Q\,(k)) \overset{\circ}{R}_Q\,(k)\| \\ &\le |h|\left(\left(1+|h|\right)\frac{1}{\varepsilon_-} + 2\right) \le 2\left(\frac{1}{\varepsilon_-} + 1\right)|h|. \end{aligned} \tag{112}$$

If we now require $2\left(\frac{1}{\varepsilon_-} + 1\right)|h| \le \frac{1}{2}$, we can use [Ka, Theorem VI.3.9] to conclude that

$$\| \overset{\circ}{R}_Q\,(k+h) - \overset{\circ}{R}_Q\,(k)\| \le 32\left(\frac{1}{\varepsilon_-} + 1\right)|h|. \tag{113}$$

Part (i) of the lemma is proved; part (ii) follows from (i) by standard arguments. □

If $\ell \in q\mathbb{Z}^3$, similar considerations apply to the operators $\overset{\circ}{M}_{\Lambda_\ell(0)}$ and $\overset{\circ}{\mathbf{M}}_{\Lambda_\ell(0)}$, which are q-periodic on the torus $\overset{\circ}{\Lambda}_\ell\,(0)$. The Floquet transform

$$\mathcal{F}_\ell : \quad L^2\left(\overset{\circ}{\Lambda}_\ell\,(0);\mathbb{C}^3\right) \to \bigoplus_{k\in\frac{2\pi}{\ell}\mathbb{Z}^3\cap\tilde{Q}} L^2\left(Q;\mathbb{C}^3\right) \tag{114}$$

is a unitary operator now defined by

$$(\mathcal{F}_\ell \Psi)(k,x) = \left(\frac{q}{\ell}\right)^{\frac{3}{2}} \sum_{m \in q\mathbb{Z}^3 \cap \breve{\Lambda}_\ell(0)} \mathrm{e}^{ik\cdot(x-m)} \Psi(x-m), \tag{115}$$

where $x \in Q$, $k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}, \Psi \in L^2\left(\overset{\circ}{\Lambda}_\ell(0); \mathbb{C}^3\right)$, $\Psi(x-m)$ being properly interpreted in the torus $\overset{\circ}{\Lambda}_\ell(0)$. We also have

$$\mathcal{F}_\ell \overset{\circ}{M}_{\Lambda_\ell(0)} \mathcal{F}_\ell^* = \bigoplus_{k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}} \overset{\circ}{M}_Q(k), \tag{116}$$

and

$$\mathcal{F}_\ell \overset{\circ}{\mathbb{S}}_{\Lambda_\ell(0)} = \bigoplus_{k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}} \overset{\circ}{\mathbb{S}}_Q(k), \quad \mathcal{F}_\ell \overset{\circ}{\mathbf{M}}_{\Lambda_\ell(0)} \mathcal{F}_\ell{}^* = \bigoplus_{k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}} \overset{\circ}{\mathbf{M}}_Q(k). \tag{117}$$

Thus we have

$$\sigma(\overset{\circ}{M}_{\Lambda_\ell(0)}) = \bigcup_{k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}} \sigma\left(\overset{\circ}{M}_Q(k)\right) \quad \text{and} \quad \sigma(\overset{\circ}{\mathbf{M}}_{\Lambda_\ell(0)}) = \bigcup_{k \in \frac{2\pi}{\ell}\mathbb{Z}^3 \cap \tilde{Q}} \sigma\left(\overset{\circ}{\mathbf{M}}_Q(k)\right). \tag{118}$$

Theorem 25 is an immediate consequence of (118) and Lemma 26.

4. Location of the Spectrum of Random Operators

In this section we prove Theorem 3. Since we already proved Theorem 25, the proof proceeds almost exactly as in [FK3, Sect. 4], so we will only outline the key steps.

In order to investigate the samples of the random quantity $\varepsilon_{g,\omega}(x)$, for a fixed g, we set

$$\mathcal{T}_g = \{\tau : \tau = \{\tau_i, i \in \mathbb{Z}^3\}, -g \le \tau_i \le g\}, \tag{119}$$

$$\mathcal{T}_g^{(n)} = \{\tau \in \mathcal{T} : \tau_{i+nj} = \tau_i \text{ for all } i,j \in \mathbb{Z}^3\}, \quad n \in \mathbb{N}, \tag{120}$$

and

$$\mathcal{T}_g^{(\infty)} = \bigcup_{n \succeq q} \mathcal{T}_g^{(n)}. \tag{121}$$

For $\tau \in \mathcal{T}_g$ we let

$$\varepsilon_\tau(x) = \varepsilon_0(x)\left[1 + \sum_{i \in \mathbb{Z}^3} \tau_i u(x-i)\right] \tag{122}$$

and

$$M(\tau) = M(\varepsilon_\tau), \quad \mathbf{M}(\tau) = \mathbf{M}(\varepsilon_\tau). \tag{123}$$

We recall (21).

To approximate Maxwell operators by periodic operators, given $\tau \in \mathcal{T}_g$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^3$, we specify $\tau_{\Lambda_n(x)} \in \mathcal{T}_g^{(n)}$ by requiring $\left(\tau_{\Lambda_n(x)}\right)_i = \tau_i$ for all $i \in \breve{\Lambda}_n(x) \cap \mathbb{Z}^3$, and define

$$M_{\Lambda_n(x)}(\tau) = M(\tau_{\Lambda_n(x)}). \tag{124}$$

The following lemma shows that the (nonrandom) spectrum of the random Maxwell operator M_g is determined by the spectra of the periodic Maxwell operators $M(\tau)$, $\tau \in \mathcal{T}_g^{(\infty)}$. The analogous result for random Schrödinger operators was proven by Kirsch and Martinelli [KM2, Theorem 4].

Lemma 27. *Let the random operator M_g defined by (28) satisfy Assumption 1, and let*

$$\Sigma_g = \overline{\bigcup_{\tau \in \mathcal{T}_g^{(\infty)}} \sigma\,(M(\tau))}. \tag{125}$$

Then $\sigma(M_g) = \Sigma_g$ with probability one.

Proof. Same proof as [FK3, Lemma 19]. $\square$

Given a real number h; $|h| < \frac{1}{U_+}$, let

$$M(h) = M(\varepsilon_h) \;\; , \mathbf{M}(h) = \mathbf{M}(\varepsilon_h) \;\text{ with }\; \varepsilon_h(x) = \varepsilon_0(x)\,[1 + hU(x)]\,. \tag{126}$$

If $|h| \le g$, and we define $\tau(h) \in \mathcal{T}_g$ by $\tau(h)_i = h$ for all $i \in \mathbb{Z}^3$, we have $\varepsilon_h = \varepsilon_{\tau(h)}$ and $M(h) = M(\tau(h))$, $\mathbf{M}(h) = \mathbf{M}(\tau(h))$.

Lemma 28. *Let $M(h)$, $|h| < \frac{1}{U_+}$, be given by (126), with ε_0 and U given in Assumption 1. Let $\Lambda = \Lambda_\ell(x_0)$ for some $x_0 \in \mathbb{R}^3$ and $\ell \succeq q$. The positive self-adjoint operator $\mathring{\mathbf{M}}(h)_\Lambda$ has compact resolvent and 0 as an eigenvalue, so let $0 < \mu_1(h) \le \mu_2(h) \le \ldots$ be its nonzero eigenvalues, repeated according to their (finite) multiplicity. Then each $\mu_j(h)$, $j = 1, 2, \ldots$, is a Lipschitz continuous, strictly decreasing function of h, with*

$$\delta_-(g) \max_{l=1,2}\{\mu_j(h_l)\} \le \frac{\mu_j(h_1) - \mu_j(h_2)}{h_2 - h_1} \le \delta_+(g) \min_{l=1,2}\{\mu_j(h_l)\} \tag{127}$$

for any $h_1, h_2 \in (-g, g)$, $0 < g < \frac{1}{U_+}$, where $\delta_\pm(g)$ are given in (27).

Proof. Same proof as [FK3, Lemma 20]. $\square$

The following corollary follows immediately from Theorem 25, Lemmas 27 and 28, and the min-max principle.

Corollary 29. *Let the random operator M_g defined by (28) satisfy Assumption 1, and let $\{\ell_n;\; n = 0, 1, 2, \ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Then*

$$\Sigma_g = \overline{\bigcup_{h \in [-g,g]} \sigma\,(M(h))} = \overline{\bigcup_{h \in [-g,g]} \bigcup_{n \ge 1} \sigma\left(\mathring{M}(h)_{\Lambda_{\ell_n}(0)}\right)}. \tag{128}$$

In particular, Σ_g is increasing in g.

Theorem 3 is now proven as in [FK3, Subsect. 4.2], using Theorem 25, Lemma 28 and Corollary 29, and taking (21) and (95) into account.

5. Dirichlet Boundary Condition for Maxwell Operators

Given an open cube Λ in $\mathbb{R}^3$ and M as in (17), we will denote by M_Λ the restriction of M to Λ with Dirichlet boundary condition, i.e., M_Λ is the nonnegative self-adjoint operator on $L^2(\Lambda;\, \mathbb{C}^3)$, uniquely defined by the nonnegative quadratic form given as the closure of

$$\mathcal{M}_\Lambda(\Psi, \Phi) = \langle \nabla \times \Psi, \frac{1}{\varepsilon} \nabla \times \Phi \rangle, \quad \Psi, \Phi \in C_0^1(\Lambda;\, \mathbb{C}^3), \tag{129}$$

the inner product being in $L^2(\Lambda;\, \mathbb{C}^3)$.

If $\varepsilon(x) \equiv 1$, we write Ξ_Λ for $M(1)_\Lambda$. Ξ_Λ has an operator core consisting of functions which are C^2 up to $\partial\Lambda$ and whose tangential component vanishes on $\partial\Lambda$. (For a discussion of boundary conditions for Maxwell operators in bounded domains see [BS].) We will need this last description to find all eigenvalues for Ξ_Λ. This is all given in the next theorem.

Some notation. If $\Psi \in C(\bar{\Lambda};\, \mathbb{C}^3)$, we use Ψ_ν and Ψ_τ to denote its (outer) normal and tangential components on $\partial\Lambda$.

Theorem 30. *Let Λ be an open cube of side L in $\mathbb{R}^3$.*

(i) *The dense linear subset*

$$\mathcal{D}_\Lambda^D = \left\{ \Psi \in C^2(\bar{\Lambda};\, \mathbb{C}^3);\ \Psi_\tau \equiv 0 \right\} \tag{130}$$

is an operator core for Ξ_Λ, with $\Xi_\Lambda \Psi = \nabla \times \nabla \times \Psi$ for $\Psi \in \mathcal{D}_\Lambda^D$.

(ii) *The operator Ξ_Λ has an orthogonal basis of eigenfunctions*

$$\begin{aligned} \Psi = \Big\{ \Psi_{\mu,j} \in \mathcal{D}_\Lambda^D;\ \mu \in \tfrac{\pi}{L}\Big(\mathbb{N}^3 \cup \big[\{0\} \times \mathbb{N}^2\big] \cup \\ \big[\mathbb{N} \times \{0\} \times \mathbb{N}\big] \cup \big[\mathbb{N}^2 \times \{0\}\big]\Big), \qquad j = 0, 1, 2 \Big\}, \end{aligned} \tag{131}$$

with

$$\nabla \times \Psi_{\mu,0} = 0, \quad \Psi_{\mu,0} = \nabla \varphi_{\mu,0} \ \textit{ with } \ \varphi_{\mu,0} \in C_0^\infty(\bar{\Lambda}); \tag{132}$$

$$\nabla \times \nabla \times \Psi_{\mu,j} = |\mu|^2 \Psi_{\mu,j}, \quad \nabla \cdot \Psi_{\mu,j} = 0, \quad j = 1, 2. \tag{133}$$

More precisely, if $\Lambda = \Lambda_L(x_0)$, we can take

$$\Psi_{\mu,j}(x) = \Phi_{\mu,j}\left(x - x_0 + \frac{L}{2}(1, 1, 1) \right), \tag{134}$$

$$\Phi_{\mu,j}(x) = \begin{bmatrix} a_1^{(\mu,j)} \cos(\mu_1 x_1) \sin(\mu_2 x_2) \sin(\mu_3 x_3) \\ a_2^{(\mu,j)} \sin(\mu_1 x_1) \cos(\mu_2 x_2) \sin(\mu_3 x_3) \\ a_3^{(\mu,j)} \sin(\mu_1 x_1) \sin(\mu_2 x_2) \cos(\mu_3 x_3) \end{bmatrix},$$

where for each $\mu \in \frac{\pi}{L}\Big(\mathbb{N}^3 \cup \big[\{0\} \times \mathbb{N}^2\big] \cup \big[\mathbb{N} \times \{0\} \times \mathbb{N}\big] \cup \big[\mathbb{N}^2 \times \{0\}\big]\Big)$ we set $a^{(\mu,0)} = \mu$ and pick $a^{(\mu,1)}, a^{(\mu,2)} \in \mathbb{R}^3$ such that $\{a^{(\mu,j)};\ j = 0, 1, 2\}$ is an orthogonal basis for $\mathbb{R}^3$.

Proof. Let the operator Γ_Λ be defined by $\Gamma_\Lambda\Psi = \nabla \times \nabla \times \Psi$ for $\Psi \in \mathcal{D}_\Lambda^D$. To see that it is a symmetric operator on its domain, notice that for $\Phi, \Psi \in C^1(\overline{\Lambda};\, \mathbb{C}^3)$ we have

$$\langle \nabla \times \Phi, \Psi \rangle - \langle \Phi, \nabla \times \Psi \rangle = \int_\Lambda \nabla \cdot (\bar{\Phi} \times \Psi) d^3x = \int_{\partial\Lambda} (\bar{\Phi} \times \Psi)_\nu dS, \tag{135}$$

where the inner products are in $L^2\left(\Lambda; \mathbb{C}^3\right)$, dS is the surface measure, and we used Gauss' Theorem. If $\Phi_\tau \equiv 0$, we must have $(\bar{\Phi} \times \Psi)_\nu \equiv 0$, so we can conclude that the surface integral in (135) equals 0.

We proceed as in [RS4, Proof of Proposition 1 in Sect. XIII.15]. To show that the symmetric operator Γ_Λ is essentially self-adjoint, it suffices to exhibit an orthogonal basis of eigenfunctions in its domain $\mathcal{D}_\Lambda^D$. Since $\left\{\cos(nx);\, n \in \frac{\pi}{L}(\{0\} \cup \mathbb{N})\right\}$ and $\left\{\sin(nx);\, n \in \frac{\pi}{L}\mathbb{N}\right\}$ are both orthogonal bases for $L^2((0, L))$, it follows that $\Psi = \{\Psi_{\mu,j}\}$, given in (134), is an orthogonal basis for $L^2(\Lambda;\, \mathbb{C}^3)$. Since

$$\Phi_{\mu,0} = \nabla[\sin(\mu_1 x_1)\sin(\mu_2 x_2)\sin(\mu_3 x_3)], \tag{136}$$

we clearly have (132). It is straightforward to check that $\Psi \subset \mathcal{D}_\Lambda^D$ and Ψ also satisfies (133), so it is an orthogonal basis of eigenfunctions for the operator Γ_Λ.

To finish the proof of the theorem, it suffices to show that Ξ_Λ is the closure $\overline{\Gamma}_\Lambda$ of Γ_Λ. To do that, notice that $C_0^2(\Lambda;\, \mathbb{C}^3) \subset \mathcal{D}_\Lambda^D \subset \mathcal{Q}(\overline{\Gamma}_\Lambda)$, where for a self-adjoint operator A we use $\mathcal{Q}(A)$ to denote the domain of the corresponding quadratic form. As quadratic forms, we clearly have $\Xi_\Lambda[\Psi] = \overline{\Gamma}_\Lambda[\Psi]$ for $\Psi \in C_0^2(\Lambda;\, \mathbb{C}^3)$, which is a form core for Ξ_Λ as a quadratic form, hence $\mathcal{Q}(\Xi_\Lambda) \subset \mathcal{Q}(\overline{\Gamma}_\Lambda)$. Since $\mathcal{D}_\Lambda^D$ is a form core for $\overline{\Gamma}_\Lambda$ as a quadratic form, to finish the proof of the theorem, it is enough to show that $\mathcal{D}_\Lambda^D \subset \mathcal{Q}(\Xi_\Lambda)$, so $\mathcal{Q}(\overline{\Gamma}_\Lambda) \subset \mathcal{Q}(\Xi_\Lambda)$.

Thus, given $\Psi \in \mathcal{D}_\Lambda^D$, it suffices to find $\Psi_n \in C_0^1(\Lambda;\, \mathbb{C}^3)$ such that

$$\|\Psi - \Psi_n\| + \|\nabla \times (\Psi - \Psi_n)\| \to 0. \tag{137}$$

Translating and scaling, if necessary, we can assume that $\Lambda = \Lambda_2(0) = (-1, 1)^3$. For each $n = 1, 2, \ldots$ we select a function $\eta_n \in C^2([-1, 1])$, $0 \le \eta_n(t) \le 1$, such that $\eta_n(t) = 1$ for $|t| \le \frac{n}{n+1}$ and $\eta_n(t) = 0$ for $\frac{n+\frac{1}{2}}{n+1} \le |t| \le 1$. We set $\Phi_n(x) = \bar{\eta}_n(x)\Theta_n(x)$, where $\bar{\eta}_n(x) = \eta_n(x_1)\eta_n(x_2)\eta_n(x_3)$ and

$$\Theta_n(x) = \begin{cases} \Psi(\frac{n+1}{n}x), & \text{if } |x_1|, |x_2|, |x_3| \le \frac{n}{n+1}; \\ \Psi(x_1, x_2, \pm 1), & \text{if } |x_1|, |x_2| \le \frac{n}{n+1},\ \frac{n}{n+1} < \pm x_3 \le 1; \\ \Psi(x_1, \pm 1, x_3), & \text{if } |x_1|, |x_3| \le \frac{n}{n+1},\ \frac{n}{n+1} < \pm x_2 \le 1; \\ \Psi(\pm 1, x_2, x_3), & \text{if } |x_2|, |x_3| \le \frac{n}{n+1},\ \frac{n}{n+1} < \pm x_1 \le 1; \\ 0, & \text{otherwise.} \end{cases} \tag{138}$$

We have $\Phi_n \in C_0(\Lambda;\, \mathbb{C}^3)$, and Φ_n is piecewise C^1 with bounded partial derivatives, so $\nabla \times \Phi_n \in L^2(\Lambda;\, \mathbb{C}^3)$. In addition,

$$\nabla \times \Phi_n = \bar{\eta}_n(\nabla \times \Theta_n) + (\nabla\bar{\eta}_n) \times \Theta_n = \bar{\eta}_n(\nabla \times \Theta_n), \tag{139}$$

since $(\nabla\bar{\eta}_n) \times \Theta_n = 0$ by our construction as $\Psi_\tau \equiv 0$. If each Φ_n was a C^1-function, instead of only piecewise C^1, we would be done, since $\Psi_n = \Phi_n$ clearly satisfies (137). To repair that we set $\Psi_n = \gamma_n * \Phi_n$, where $\{\gamma_n\}$ is a suitably chosen approximate identity, i.e., $\gamma_n(x) = n^3\gamma(nx)$ for some positive C^∞ function γ on $\mathbb{R}^3$ with support on $\Lambda_1(0)$ and $\int \gamma(x)dx = 1$, so $\Psi_n \in C_0^1(\Lambda; \mathbb{C}^3)$, $\nabla \times \Psi_n = \gamma_n * (\nabla \times \Psi_n)$, and (137) is satisfied. $\square$

The Weyl decomposition corresponding to Dirichlet boundary condition is given by $L^2(\Lambda; \mathbb{C}^3) = \mathbb{S}_\Lambda \oplus \mathbb{G}_\Lambda$, where $\mathbb{G}_\Lambda$ and $\mathbb{S}_\Lambda$ are the closed subspaces spanned by $\{\Psi_{\mu,0}\}$ and $\{\Psi_{\mu,j},\ j = 1, 2\}$, respectively, where $\{\Psi_{\mu,j},\ j = 0, 1, 2\}$ is the orthogonal basis given in (131). It is easy to see that

$$\mathbb{G}_\Lambda = \overline{\{\Psi \in C_0^1(\Lambda; \mathbb{C}^3);\ \ \Psi = \nabla\varphi \ \text{ with } \ \varphi \in C_0^1(\Lambda)\}}, \tag{140}$$

$$\mathbb{S}_\Lambda = \{\Psi \in L^2(\Lambda; \mathbb{C}^3);\ \ \nabla \cdot \Psi = 0 \ \text{ weakly } \} \tag{141}$$

The spaces $\mathbb{S}_\Lambda$ and $\mathbb{G}_\Lambda$ are left invariant by M_Λ, with $\mathbb{G}_\Lambda \subset \mathcal{D}(M_\Lambda)$ and $M_\Lambda|_{\mathbb{G}_\Lambda} = 0$. We define $\mathbf{M}_\Lambda$ as the restriction of M_Λ to $\mathbb{S}_\Lambda$, i.e., $\mathcal{D}(\mathbf{M}_\Lambda) = \mathcal{D}(M_\Lambda) \cap \mathbb{S}_\Lambda$ and $\mathbf{M}_\Lambda = M_\Lambda|_{\mathcal{D}(M_\Lambda)\cap\mathbb{S}_\Lambda}$. Notice $M_\Lambda = \mathbf{M}_\Lambda \oplus 0_{\mathbb{G}_\Lambda}$, $0 \notin \sigma(\mathbf{M}_\Lambda)$, so $\sigma(\mathbf{M}_\Lambda) = \sigma(M_\Lambda)\backslash\{0\}$.

$\mathbf{M}_\Lambda$ and M_Λ will be called Dirichlet Maxwell operators. We write Ξ_Λ for $\mathbf{M}_\Lambda(1)$. Notice that Ξ_Λ is a strictly positive operator with discrete spectrum; the same being true of $\mathbf{M}_\Lambda$ in view of (14).

Corollary 31. *Let M be as in (17) with (14), and let Λ be an open cube in $\mathbb{R}^3$. Then*

(i) $\mathbf{M}_\Lambda$ *has compact resolvent; in fact* $\operatorname{Tr}\{(\mathbf{M}_\Lambda + I)^{-p}\} < \infty$ *for any* $p > \frac{3}{2}$.
(ii) *For any $E > 0$ let $n_{\varepsilon,\Lambda}(E)$ denote the number of eigenvalues of $\mathbf{M}_\Lambda$ less than E, each eigenvalue counted as many times as its multiplicity. There exists a finite constant C_0, independent of Λ and ε, such that*

$$n_{\varepsilon,\Lambda}(E) \leq C_0\varepsilon_+^{\frac{3}{2}}|\Lambda|E^{\frac{3}{2}}. \tag{142}$$

Proof. We clearly have $\mathbf{M}_\Lambda \geq \frac{1}{\varepsilon_+}\Xi_\Lambda$, so it suffices to prove the corollary for Ξ_Λ.

It follows from Theorem 30(ii) that the spectrum of Ξ_Λ consists of eigenvalues whose multiplicity can be read from (131), so an explicit calculation gives $\operatorname{Tr}\{(\Xi_\Lambda + I)^{-p}\} < \infty$ for any $p > \frac{3}{2}$. A similar calculation gives (142). $\square$

Remark 32. $n_{\varepsilon,\Lambda}(E)$ is also equal to the number of *strictly positive* eigenvalues of M_Λ less than E, each eigenvalue counted as many times as its multiplicity.

6. A Wegner-Type Estimate

Given an open cube Λ in $\mathbb{R}^3$, we will denote by $M_{g,\Lambda} = M_{g,\omega,\Lambda}$ the restriction of the random operator $M_{g,\omega}$ to Λ with Dirichlet boundary condition. Notice that $M_{g,\omega,\Lambda}$ is a random operator on $L^2(\Lambda)$, measurability follows from [FK3, Theorem 38]. Each $\mathbf{M}_{g,\omega,\Lambda}$ has compact resolvent by Corollary 31(i). For any $E > 0$ we define $n_{g,\Lambda}(E) = n_{g,\omega,\Lambda}(E)$ as the number of *strictly positive* eigenvalues of $M_{g,\omega,\Lambda}$ less than E. Notice that $n_{g,\omega,\Lambda}(E)$ is the distribution function of the measure $n_{g,\omega,\Lambda}(dE)$ on $(0, \infty)$ given by

$$\int h(E)n_{g,\omega,\Lambda}(dE) = \operatorname{Tr}(h(M_{g,\omega,\Lambda})) = \operatorname{Tr}(h(\mathbf{M}_{g,\omega,\Lambda})) \tag{143}$$

for positive continuous functions h with compact support in $(0, \infty)$.

We will say that the random operator M_g defined by (28) satisfies Assumption 1′, if it satisfies all of Assumption 1 with the exception of the requirement that $\varepsilon_0(x)$ be a q -periodic function.

We have the following “a priori” estimate, which is an immediate consequence of Corollary 31(ii), (26) and Assumption 1(iv) .

Lemma 33. *Let the random operator M_g defined by (28) satisfy Assumption 1′. There exists a finite constant C_1, depending only on $\varepsilon_{0.+}$, such that we have*

$$n_{g,\omega,\Lambda}(E) \le C_1 |\Lambda| E^{\frac{3}{2}} \tag{144}$$

for all $\omega \in [-1,1]^{\mathbb{Z}^3}$, for all $E > 0$ and all open cubes Λ in $\mathbb{Z}^3$.

Theorem 34 (Wegner-type estimate). *Let the random operator M_g defined by (28) satisfy Assumption 1′. There exists a constant $Q < \infty$, depending only on the constants r_u and $\varepsilon_{0,+}$, such that*

$$\mathbb{P}\{\operatorname{dist}(\sigma(M_{g,\omega,\Lambda}), E) \le \eta\} \le Q \frac{U_- + 2U_+}{gU_+(1 - gU_+)U_-} \|\rho\|_\infty |E|^{\frac{1}{2}} \eta |\Lambda|^2 \tag{145}$$

for all $E > 0$, open cubes Λ in $\mathbb{R}^3$, and all $\eta \in [0, E)$.

Proof. The proof is exactly the same as the proof of [FK3, Theorem 23], with the proviso that we only integrate $n_{g,\omega,\Lambda}(E)$ against positive continuous functions with compact support in $(0,\infty)$. $\square$

7. Localization

Theorems 6 and 7 are proved exactly as in [FK3], applying a multiscale analysis appropriate for random perturbations of periodic operators on $\mathbb{R}^3$ [FK3, Theorems 29 and 35] to operators M_g as in (28).

Let the operator M be as in (17) with (14). Given an open cube Λ in $\mathbb{R}^3$, M_Λ is the restriction of M to Λ with Dirichlet boundary condition (see Sect. 5). Each M_Λ is a nonnegative self-adjoint operator on $L^2(\Lambda; \mathbb{C}^3)$ with compact resolvent $R_\Lambda(z) = (M_\Lambda - z)^{-1}$. If $\Lambda = \Lambda_L(x)$, we will write $M_{x,L} = M_{\Lambda_L(x)}$ and $R_{x,L}(z) = R_{\Lambda_L(x)}(z)$. The norm in $L^2(\Lambda; \mathbb{C}^3)$ and also the corresponding operator norm will both be denoted by $\| \ \|_{x,L}$. If $\Lambda_1 \subset \Lambda_2$ are open cubes, $J_{\Lambda_1}^{\Lambda_2} : L^2(\Lambda_1; \mathbb{C}^3) \to L^2(\Lambda_2; \mathbb{C}^3)$ is the canonical injection. If $\Lambda_i = \Lambda_{L_i}(x_i)$, $i = 1, 2$, we write $\| \ \|_{x_1,L_1}^{x_2,L_2}$ for the (operator) norm in $\mathcal{B}\left(L^2(\Lambda_{L_1}(x_1); \mathbb{C}^3), L^2(\Lambda_{L_2}(x_2); \mathbb{C}^3)\right)$ and $J_{x_1,L_1}^{x_2,L_2} = J_{\Lambda_{L_1}(x_1)}^{\Lambda_{L_2}(x_2)}$. If $\varphi \in L^\infty(\Lambda)$, we also use φ to denote the operator on $L^2(\Lambda; \mathbb{C}^3)$ given by multiplication by φ; if $\Phi \in L^\infty(\Lambda; \mathbb{C}^3)$ we write $\Phi^\times$ for the operator $\Phi\times$, i.e., $\Phi^\times \Psi = \Phi \times \Psi$.

7.1. The basic technical tools. The results of [FK3, Subsects. 6.1 and 6.3] are valid for the Maxwell operator M, with the obvious modifications. We state the key results for completeness. We start with the *smooth resolvent identity* (SRI), which is used to relate resolvents in different scales.

Lemma 35 (SRI). *Let the operator M be given by (17) with (14), let $\Lambda_1 \subset \Lambda_2$ be open cubes in $\mathbb{R}^3$, and let $\varphi_1 \in C_0^1(\Lambda_1)$. Then, for any $z \notin \sigma(M_{\Lambda_1}) \cup \sigma(M_{\Lambda_2})$ we have*

$$R_{\Lambda_2}(z) J_{\Lambda_1}^{\Lambda_2} \varphi_1 = \tag{146}$$
$$J_{\Lambda_1}^{\Lambda_2} \varphi_1 R_{\Lambda_1}(z) + R_{\Lambda_2}(z) \left(-J_{\Lambda_1}^{\Lambda_2} (\nabla \varphi_1)^{\times} \frac{1}{\varepsilon} \nabla^{\times} + \nabla^{\times} J_{\Lambda_1}^{\Lambda_2} \frac{1}{\varepsilon} (\nabla \varphi_1)^{\times} \right) R_{\Lambda_1}(z)$$

as quadratic forms on $L^2(\Lambda_2; \mathbb{C}^3) \times L^2(\Lambda_1; \mathbb{C}^3)$.

Proof. The lemma follows immediately from [FK3, Lemma 24] and the definition of Dirichlet boundary condition. $\square$

To take into account the periodicity of the background medium, $q \in \mathbb{N}$ being the period (see Assumption 1), we work with boxes $\Lambda_L(x)$ with $x \in q\mathbb{Z}^3$ and $L \in 2q\mathbb{N}$, so the background is the same in all boxes in a given scale L. For such boxes (with $L \geq 4q$) we set

$$\Upsilon_L(x) = \{ y \in q\mathbb{Z}^3;\ \ \|y - x\| = \frac{L}{2} - q \} \tag{147}$$

and

$$\tilde{\Upsilon}_L(x) = \Lambda_{L-q}(x) \backslash \overline{\Lambda}_{L-3q}(x), \ \ \hat{\Upsilon}_L(x) = \Lambda_{L-\frac{3q}{2}}(x) \backslash \overline{\Lambda}_{L-\frac{5q}{2}}(x). \tag{148}$$

We also set

$$\chi_x = \chi_{x,q} \ \text{ and } \ \Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)}, \ \ \hat{\Gamma}_{x,L} = \chi_{\hat{\Upsilon}_L(x)}. \tag{149}$$

Notice

$$\Gamma_{x,L} = \sum_{y \in \Upsilon_L(x)} \chi_y \ \text{ a.e.} \tag{150}$$

and

$$|\Upsilon_L(x)| \leq 3(L - 2q + 1)^2. \tag{151}$$

In addition each $\Lambda_L(x)$ will be equipped with a function $\Phi_{x,L}$ constructed in the following way: we fix an even function $\xi \in C_0^1(\mathbb{R})$ with $0 \leq \xi(t) \leq 1$ for all $t \in \mathbb{R}$, such that $\xi(t) = 1$ for $|t| \leq \frac{q}{4}$, $\xi(t) = 0$ for $|t| \geq \frac{3q}{4}$, and $|\xi'(t)| \leq \frac{3}{q}$ for all $t \in \mathbb{R}$. (Such a function always exists.) We define

$$\xi_L(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{L}{2} - \frac{5q}{4} \\ \xi\left(|t| - \left(\frac{L}{2} - \frac{3q}{2} \right) \right), & \text{if } |t| \geq \left(\frac{L}{2} - \frac{3q}{2} \right) \end{cases} \tag{152}$$

and set

$$\Phi_{x,L}(y) = \Phi_L(y - x) \ \text{ for } \ y \in \mathbb{R}^3, \ \text{ with } \ \Phi_L(y) = \prod_{i=1}^{3} \xi_L(y_i). \tag{153}$$

We have $\Phi_{x,L} \in C_0^1(\Lambda_L(x))$, $0 \leq \Phi_{x,L} \leq 1$,

$$\chi_{x, \frac{L}{2} - \frac{5q}{4}} \Phi_{x,L} = \chi_{x, \frac{L}{2} - \frac{5q}{4}}, \quad \chi_{x, \frac{L}{2} - \frac{3q}{4}} \Phi_{x,L} = \Phi_{x,L}, \tag{154}$$

and

$$\hat{\Gamma}_{x,L} \left(\nabla \Phi_{x,L} \right) = \nabla \Phi_{x,L}, \quad |\nabla \Phi_{x,L}| \leq \frac{3\sqrt{3}}{q}. \tag{155}$$

We can now state a *Simon-Lieb-type inequality* (SLI); it is used to obtain decay in a larger scale from decay in a given scale.

Lemma 36 (SLI). *Let the operator M be given by (17) with (14). Then for any $\ell, L \in 2q\mathbb{N}$ with $4q \le \ell < L - 3q$, $x, y \in q\mathbb{Z}^3$ with $2\|y - x\| \le L - \ell - 3q$ (so $\Lambda_\ell(y) \subset \Lambda_{L-3q}(x)$), and $z \notin \sigma(M_{x,L}) \cup \sigma(M_{y,\ell})$, we have*

$$\|\Gamma_{x,L} R_{x,L}(z)\chi_y\|_{x,L} \le \gamma_z \ell^2 \|\Gamma_{y,\ell} R_{y,\ell}(z)\chi_y\|_{y,\ell} \|\Gamma_{x,L} R_{x,L}(z)\chi_{y'}\|_{x,L} \tag{156}$$

for some $y' \in \Upsilon_{y,\ell}$, with

$$\gamma_z = \frac{18\sqrt{3}}{q\varepsilon_-} \Theta_{\frac{q}{4}} \sqrt{\varepsilon_+} \left(\sqrt{\varepsilon_-} + \frac{1}{\sqrt{\varepsilon_-}} \right) (1 + |z|), \tag{157}$$

where $\Theta_{\frac{q}{4}}$ is the constant given in Corollary 12.

Proof. The lemma is proved as [FK3, Lemma 26], using Lemma 35 and Corollary 12. □

The *eigenfunction decay inequality* (EDI) is used to obtain decay for generalized eigenfunctions from decay of local resolvents.

Lemma 37 (EDI). *Let the operator M be given by (17) with (14), and let Ψ be a generalized eigenfunction for a given $z \in \mathbb{C}$. For any $x \in q\mathbb{Z}^3$ and $\ell \in 2q\mathbb{N}$ with $\ell \ge 4q$, such that $z \notin \sigma(M_{x,\ell})$, we have*

$$\|\chi_x \psi\| \le \gamma_z \ell^2 \|\Gamma_{x,\ell} R_{x,\ell}(z^*)\chi_x\|_{x,\ell} \|\chi_y \psi\| \tag{158}$$

for some $y \in \Upsilon_{y,\ell}$, with γ_z as in (157).

Proof. Same proof as [FK3, Lemma 27]. □

The starting hypothesis for the multiscale analysis [FK3, (P1) in Theorem 29 and (H1) in Theorem 35] is formulated for operators with Dirichlet boundary condition. But under the hypotheses of Theorems 6 and 7 the natural starting hypothesis is the analogue of either (P1) or (H1) for *periodic* boundary condition. The following lemma enable us to go from periodic boundary condition to Dirichlet boundary condition.

For M_g be as in (28) satisfying Assumption 1, $x \in q\mathbb{Z}^3$ and $L \in 2q\mathbb{N}$, we set (with the notation of (124))

$$\overset{\circ}{M}_{g,\omega,x,L} = (\overset{\circ}{M}((g\omega)_{\Lambda_L(x)}))_{\Lambda_L(x)}, \tag{159}$$

which is a random operator by [FK3, Theorem 38]. We write $\overset{\circ}{R}_{g,\omega,x,L}(z)$ for its resolvent.

Lemma 38. *Let M_g be as in (28) satisfying Assumption 1. Let $E > 0$, $x \in q\mathbb{Z}^3$ and $L \in 2q\mathbb{N}$, $L \ge 4q$; set $\hat{L} = L + [2r_u]_{2q} + 2q$. If ω is such that $E \notin \sigma(M_{g,\omega,x,L} \cup \sigma(\overset{\circ}{M}_{g,\omega,x,\hat{L}})$, then*

$$\begin{aligned} &\|\Gamma_{x,L} R_{g,\omega,x,L}(E)\chi_x\|_{x,L} \le \\ &\left(1 + \frac{3\sqrt{3}}{q\varepsilon_-}\left(1 + 2(1+E)\|R_{g,\omega,x,L}(E)\|_{x,L}\right)\right) \|\Gamma_{x,L} \overset{\circ}{R}_{g,\omega,x,\hat{L}}(E)\chi_x\|_{x,\hat{L}}. \end{aligned} \tag{160}$$

Proof. Same proof as [FK3, Lemma 37]. □

7.2. The proofs of localization. Theorems 6 and 7 can now be proved exactly as in [FK3], using Theorems 3, 25, 34, and Lemmas 24, 27, 36, 37, 38, so we refer the reader to [FK3, Subsects. 6.4 and 6.5].

Acknowledgement. Effort sponsored by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant F49620-94-1-0172, and by the Division of Mathematical Sciences of the National Science Foundation, under grant DMS-9500720. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research, the National Science Foundation, or the US Government.

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Communicated by A. Kupiainen