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BAND-GAP STRUCTURE OF SPECTRA OF PERIODIC DIELECTRIC AND ACOUSTIC MEDIA. I. SCALAR MODEL

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Abstract. We investigate the band-gap structure of the spectrum of second-order partial differential operators associated with the propagation of waves in a periodic two-component medium. The medium is characterized by a real-valued position-dependent periodic function $\varepsilon(x)$ that is the dielectric constant for electromagnetic waves and mass density for acoustic waves. The imbedded component consists of a periodic lattice of cubes where $\varepsilon(x) = 1$. The value of $\varepsilon(x)$ on the background is assumed to be greater than 1. We give the complete proof of existence of gaps in the spectra of the corresponding operators provided some simple conditions imposed on the parameters of the medium.

Key words: propagation of electromagnetic and acoustic waves, band-gap structure of the spectrum, periodic dielectrics, periodic acoustic media.

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1. INTRODUCTION. One of the main observations in the quantum theory of solids is that the energy spectrum of an electron in a solid consists of bands separated by gaps (see, for instance, [AM]). This band-gap structure arises due to the periodicity of the underlying crystal. Such a structure of the spectrum is common for many periodic differential operators (see the so-called Floquet-Bloch theory in [E],[K93],[RS]). It is natural to ask whether the same kind of phenomenon can occur for classical electromagnetic and acoustic waves provided that the underlying nonhomogeneous medium is periodic. It is not hard to show that the answer is positive (see the subsection about direct integral decomposition below). However, such results only show that gaps *may exist in principle*; the practically important question is whether they really exist in concrete situations (if two bands of the spectrum overlap, then the corresponding gap disappears). The idea of finding and designing periodic dielectric materials that exhibit gaps in the spectrum was introduced in [Y], [J87]. The basic physical reason for the rise of gaps lies in the coherent multiple scattering and interference of waves (see, for instance, S. John [J91] and references therein). The tremendous number of applications which are expected in optics and electronics (including high efficiency lasers, laser diodes, etc.) warrant the thorough investigation of this matter. So, one is not surprised by the persistent attention that this problem has attracted. The most recent theoretical and experimental results on the photonic band-gap structures are published in the series of papers [DE]. Some approaches to a theoretical treatment of two-dimensional (2D) periodic dielectrics were developed in [VP] for sinusoidally and rectangularly modulated

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dielectric constants, and in [PM], [MM] for a periodic array of parallel dielectric rods of circular cross section, whose intersections with a perpendicular plane form a triangular or square lattice. The experimental results (see [vAL],[DG],[YG]) for periodic and disordered dielectrics indicate that the photonic gap regime can be achieved for some nonhomogeneous materials. Analysis and numerics of some approximate models (see [JR], [EZ], [LL], [ZS], [HCS]) have shown the possibility of a gap (or pseudogap) regime for some two-component periodic dielectrics. The list of publications on the subject is already rather lengthy and we do not intend to present the complete bibliography.

One of the main obstacles in the theoretical treatment of the problem is the lack of nontrivial multidimensional models of nonhomogeneous media that can constructively explain under what circumstances the gaps arise, and how to design materials with gaps in a desired region. In particular, there has been no rigorous proof, or even nonrigorous analytic arguments (at least, we do not know one), of the existence of the gaps for $2D$ or $3D$ periodic media, the consideration of which cannot be reduced easily to the one-dimensional case. The main purpose of this and of subsequent papers is to provide some rigorous mathematical approaches that enable one to treat the problem both analytically and numerically. We prove existence of gaps under some simple conditions on the medium. The consideration is restricted to the case of two-component dielectric and acoustic media only. There are two reasons for this. First, two-component media are the simplest nonhomogeneous media which can be analyzed rigorously (including explicit relations between the parameters of the media and the structure of the spectra). Second, fabrication of this kind of media is more feasible. The basic idea of our approach was outlined in [FK]. We consider a medium that consists of a periodic array of air cubes imbedded in an optically dense host material. The method is based on analysis of the relevant boundary value problem for second-order differential operators. We assume that the dielectric constant contrast (or the mass density contrast in the acoustic case) tends to infinity, and the distance between the air cubes tends to zero. We show that if the rates of these two convergencies are properly related to each other, then one can guarantee the existence of gaps in some prescribed parts of the spectrum. If there is no such coordination between the two rates, then one should expect the rise of pseudogaps rather than of real gaps [F94] (here pseudogaps mean parts of the spectrum where the spectrum is "thin" in some sense).

The relevant mathematical problem consists in investigating the spectral properties of some self-adjoint second-order differential operators. Such an operator for electromagnetic waves has the following form:

$$\Lambda\Psi = \nabla \times (\gamma(x)\nabla \times \Psi), \nabla \cdot \Psi = 0, \gamma(x) = \varepsilon^{-1}(x), x \in \mathbf{R}^3, \tag{1}$$

where $\Psi(x)$ is a complex vector function on $\mathbf{R}^3$, and $\varepsilon(x)$ stands for the electric permittivity. An important scalar analog of this operator is:

$$\Gamma\psi = -\sum_{j=1}^{D} \frac{\partial}{\partial x_j}\gamma(x)\frac{\partial}{\partial x_j}\psi, x \in \mathbf{R}^D,\ D = 3, 2 \tag{2}$$

It can be associated with propagation of acoustic waves for $D = 3$, and $\varepsilon(x) = \gamma(x)^{-1}$ is the mass density of the medium. We assume that the coefficient $\varepsilon(x), x \in \mathbf{R}^3$ is a periodic function bounded from above and below by some positive constants. The important parameters of such a two-component periodic medium [YG] are the volume-filling fraction, the dielectric constant contrast $\varepsilon_b/\varepsilon_a$ (where ε_b and ε_a are, respectively, the dielectric constants of the host material and the embedded components), and the shape of atoms of the embedded material as well as their arrangement. In particular, high dielectric constant contrast favors the rise of gaps in the spectrum (some living tissues possess very high contrast [P]).

Our analysis shows existence of gaps for two-component dielectrics under certain conditions. The existence of gaps can be proved fairly easily for the lattice (finite-difference) version of the relevant operators, i.e., for some kind of Anderson model of electromagnetism or acoustics [F93]. In addition to that, the limit location of the bands of the spectrum was found for both discrete and continuous models in [F93] and [F94] under the assumption that the dielectric constant contrast between the background and the embedded component is large. This limit spectrum is just the (discrete) spectrum of the Neumann Laplacian in the single cubic "atom" of air. The lattice model case is much simpler than the continuous one, due to boundedness of corresponding difference versions of the original differential operators. In other words, in dealing with lattice models we cut off high frequencies. Unfortunately, the arguments used in the discrete case are not transferable to the continuous models. We have developed an alternative approach to continuous models based on Floquet-Bloch theory [E],[K82],[K93],[RS] and on variational methods for the relevant quadratic forms. Our analysis shows that to open up a gap at a preassigned point of the spectrum, one must satisfy certain quite simple relationships between the geometric parameters of the medium and the dielectric constant contrast

In this paper we investigate the spectrum of the operator Γ, i.e., the scalar case. This operator is associated with the following physically important cases:

(i) In the three dimensional case the operator Γ governs the propagation of acoustic waves in a medium with periodically varying density.

(ii) In two dimensions the operator Γ describes propagation of electromagnetic waves (namely, so-called H polarized mode) in a periodic medium that consists of a periodic array of parallel rods with square cross section and dielectric constant $\varepsilon = 1$ embedded into a background material with higher dielectric constant; this operator arises if one investigates the operator Λ for electromagnetic waves propagating in directions perpendicular to the rods.

The Maxwell operator Λ for a two-component dielectric medium consisting of a periodic array of cubes with dielectric constant $\varepsilon = 1$ embedded in a background material with higher dielectric constant has spectral structure similar to the spectrum of the operator Γ. We will provide the proof for this case in the next publication.

The following comments might be helpful in reading the paper.

1. The one-dimensional version of the operator Γ turns out to be a very good guide for multidimensional operators Γ and Λ.

2. The main idea of our approach consists of reducing the original spectral problem to a small perturbation of the spectral problem for the Neumann Laplacian in the "air bubble" cube (this explains why the spectrum concentrates in the vicinity of the Neumann spectrum). This idea was clearly expressed in [FK]. We provide an alternative proof here, but the underlying idea remains the same.

3. We consider the asymptotic case when the dielectric contrast (or the mass contrast in the acoustic case) approaches infinity, whereas the distance between the air cubes approaches zero (with a rate coordinated with the contrast one). The limit spectrum is the spectrum of the Neumann Laplacian in the single air cube.

The paper is organized as follows. In the next section we describe the medium and formulate the main statements. Some necessary constructions of Floquet-Bloch theory are provided in the section 3. We would like to mention that nonsmoothness of the boundary of the air bubbles creates known difficulties: solutions of the corresponding diffraction problem do not have the "correct" smoothness. There are three possible ways of overcoming this problem (and we believe that all of them are equally applicable here): either one describes exact domains of the corresponding operators, using the known techniques for elliptic problems in nonsmooth domains [G], or one smooths off small corners of the cubic air bubbles, or one avoids using the operators, working with quadratic forms only. We have chosen the third approach, which we found to be the simplest one. The standard Floquet theory, however, has not been explicitly developed in terms of quadratic forms. This is why we must include a large section devoted to the corresponding Floquet theory. Then, we consider the one-dimensional case. It is used later for the estimates from above for the eigenvalues. The next section contains estimates from below for the eigenvalues. Finally, we prove the main theorem.

2. STATEMENT OF THE MAIN RESULT. We begin with the description of the medium. Since the three-dimensional case is technically more complicated, we provide detailed arguments and notation for this case. In the two-dimensional case the relevant arguments are the same (in fact, even simpler). Some multidimensional generalizations are possible (and straightforward).

Throughout the paper we will use three important parameters: $\delta \in (0, \frac{1}{2})$,$\gamma \in (0, 1)$, and $\varepsilon = \gamma^{-1}$. The meaning of these parameters will be clear from the context. Let us denote by

$$X = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : 0 \leq x_j \leq 1, 1 \leq j \leq 3\}$$

the unit cube in $\mathbf{R}^3$. This will be our main cell of periods. We will also use the smaller cube (which will be filled with air, i. e. $\varepsilon(x) = 1$ in this cube)

$$O_\delta = \{x \in \mathbf{R}^3 : \delta \leq x_j \leq 1\}$$

so the parameter δ characterizes the distance between the boundaries of these two cubes. The boundary of O_δ is denoted by G_δ. The complement to O_δ in X is $R_\delta = X - O_\delta$ (this portion of the space is filled with optically dense material). We will often need the three-dimensional torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$. Since the natural projection $\mathbf{R}^3 \to \mathbf{T}^3$ is bijective on O_δ, we will identify the domain O_δ with its image in this torus. The union of all translations of O_δ by elements of $\mathbf{Z}^3$ is denoted by U_δ, and the corresponding union for R_δ is denoted by V_δ, so $U_\delta \cup V_\delta = \mathbf{R}^3, U_\delta \cap V_\delta = \emptyset$. The complement to O_δ in $\mathbf{T}^3$ is T_δ, i.e. $T_\delta = V_\delta/\mathbf{Z}^3 = R_\delta/\mathbf{Z}^3 \subset \mathbf{T}^3$. The dual lattice to $\mathbf{Z}^3$ is $2\pi\mathbf{Z}^3$, and we denote by K its standard fundamental domain:

$$K = \{k = (k_1, k_2, k_3) \in \mathbf{R}^3 : 0 \le k_j \le 2\pi, 1 \le j \le 3\}.$$

We shall have a lot of constants involved in estimates below. So we adopt the following convention on the notation for constants. We denote constants by $C_{a,b,..}$, where indices indicate parameters that determine the values of the constant C . Note that according to our convention, $C_{a,b,..}$ can have different numerical values in different formulas! We adopt the notation C for absolute constants (which can have different numerical values in different formulas).

We also use the following notation:

$L^p(Y)$ is the standard Lebesque space of functions on Y integrable in p

$H^p(Y)$ is the standard Sobolev space of order p on Y (see [EE]);

D_i denotes the partial derivative $\frac{\partial}{\partial x_i}$;

$D^\alpha = D_1^{a_1}...D_n^{a_n}$;

$\overline{O}$ is the closure of a set O ;

$\overline{c}$ is the conjugate number to a complex number c;

For a linear operator or a quadratic form Q we denote its domain by $\mathcal{D}(Q)$.

To formulate the main statement we shall also need the spectrum $\nu_0 \le \nu_1 \le \dots$ of the Neumann problem in the unit cube , i.e.

$$(3) \qquad \{\nu_j, j \ge 0\} = \{(\pi n)^2, n \in \mathbf{Z}^3(\text{ or } \ \mathbf{Z}^2 \ \text{ in the two dimensional case })\}$$

In other words, we use symbols ν_j for both $2D$ and $3D$ cases. It will be clear from the context which one is used .

We now formulate the main result of the paper. We use the notation $\Gamma_{\delta,\gamma}$ for our main operator. It is rigorously defined in the next section, and so far the reader can think of it as of a proper realization in $L^2(\mathbf{R}^3)$ of the operator (2), where $\gamma(x)$ is equal to 1 on U_δ and is equal to a constant γ on V_δ. The following statement represents our main result. (As was mentioned before, it holds both in three and two dimensions, so we restrict ourselves to the three-dimensional case.)

THEOREM 2.1. *Let N be a positive number, and ν_m be the largest eigenvalue of the Neumann Laplacian in O_δ that is smaller than N. Then, for sufficiently small values of $\delta, \delta^{-1}\gamma$ and $\gamma^{-1}\delta^2$ the part of the spectrum $\sigma(\Gamma_{\delta,\gamma})$ that belongs to the interval $[0, N]$ satisfies the following inclusion:*

$$(4) \qquad (\sigma(\Gamma_{\delta,\gamma}) \bigcap [0, N]) \subset \bigcup_{j \le m} I_j \ , I_j = [(1 - \alpha_N)\nu_j, \nu_j + C\gamma\delta^{-1}],$$

where C is a constant, and for some constants A and B

$$\alpha_N = A\delta + \max\{B\delta, C\delta^2\gamma^{-1}\}N$$

In addition to that,

$$\sigma(\Gamma_{\delta,\gamma})\bigcap I_j \neq \emptyset \ \textit{ for all } \ j \leq m$$

This theorem shows, in particular, that for sufficiently small values of $\delta, \delta^{-1}\gamma$ and $\gamma^{-1}\delta^2$ we can create gaps in any prescribed finite part of the spectrum, since it concentrates in a vicinity of the Neumann spectrum. This theorem also states what the width of bands of the spectrum is.

Corollary 2.2. *Under the notation of the Theorem 2.1, for any nonnegative m there exists a positive constant C_m such that for $\delta = \varepsilon^{-2/3}$ the lengths $|I_j|$ of the intervals I_j can be estimated as*

$$|I_j| \leq C_m \varepsilon^{-1/3}, \ \varepsilon > 1, 1 \leq j \leq m \tag{5}$$

One proves this corollary by simply plugging $\delta = \varepsilon^{-2/3}$ into the definition of I_j. The value $\delta = \varepsilon^{-2/3}$ arises, if one wants to minimize the expression

$$\max\{\delta^2\varepsilon, (\varepsilon\delta)^{-1}\}.$$

The rest of the paper is devoted to the proof of Theorem 2.1.

3. QUADRATIC FORMS AND DIRECT INTEGRAL DECOMPOSITIONS. In this section we extend some construction of Floquet-Bloch theory to the case of the operator Γ. The choice of the fundamental domain X of the group of periods is irrelevant for our arguments. In this section it will be convenient to use the domain $X : \delta/2 \leq x_j \leq 1+\delta/2$ whose boundary does not touch the surfaces of the discontinuity of ε. In the following sections we shall return to the original definition of X. We will often employ the well-known connection between nonnegative self-adjoint operators and quadratic forms. Particularly we shall need the following statement (see [D],[RS]).

Proposition 3.1. *If $\mathcal{H}$ is a Hilbert space and $Q(f,g)$ is a nonnegative closed form on it with the domain $\mathcal{D}_Q$, then there exists a nonnegative self-adjoint operator q such that $\mathcal{D}(\sqrt{q}) = \mathcal{D}_Q$ and $Q(f,g) = (f, qg)$ for any $f \in \mathcal{D}_Q$ and $g \in \mathcal{D}(q)$.*

The following consequence of the proposition holds.

Corollary 3.2. *Let Q and q be a nonnegative quadratic form and the corresponding self-adjoint operator, and $f \in \mathcal{D}_Q$ be a vector. If the following identity holds*

$$Q(f,g) = (h,g), \forall g \in \mathcal{D}_Q, \tag{6}$$

then f belongs to $\mathcal{D}(q)$, and $h = qf$.

Proof. From the proposition and (6) we get

$$(\sqrt{q}f, \sqrt{q}g) = (h,g), \forall g \in \mathcal{D}_Q = \mathcal{D}(\sqrt{q}).$$

Since $\sqrt{q}$ is self-adjoint, this implies that $\sqrt{q}f \in \mathcal{D}(\sqrt{q})$, which in turn implies that $f \in \mathcal{D}(q)$. This completes the proof of the corollary.

We now define the following nonnegative quadratic form in $L_2(\mathbf{R}^3)$ with domain equal to the Sobolev space $H^1(\mathbf{R}^3)$:

$$Q_{\delta,\gamma}(f,g) = \int\limits_{U_\delta} \nabla f \cdot \overline{(\nabla g)}\, dx + \gamma \int\limits_{V_\delta} \nabla f \cdot \overline{(\nabla g)}\, dx, Q[f] = Q(f,f)$$

LEMMA 3.3. *The form $Q_{\delta,\gamma}$ is closed.*

Proof. The square of the H^1-norm of a function f is equivalent to the sum $||f||^2_{L_2} + Q_{\delta,\gamma}(f,f)$, and the space H^1 is complete.□

In view of Proposition 3.1, there exists a self-adjoint operator $\Gamma_{\delta,\gamma} \geq 0$ in $L_2(\mathbf{R}^3)$ with the domain $\mathcal{D}_{\delta,\gamma}$ such that

$$Q_{\delta,\gamma}[f] = (\Gamma_{\delta,\gamma} f, f), \text{ for all } f \in \mathcal{D}_{\delta,\gamma}$$

Let $p \in \mathbf{Z}^3$. We introduce the shift operator A_p on $L_2(\mathbf{R}^3)$ as follows: $(A_p f)(x) = f(x+p)$. It is isometric both in $L_2(\mathbf{R}^3)$ and $H^1(\mathbf{R}^3)$, and all these operators form the commutative group $\mathcal{T} = \{A_p | \, p \in \mathbf{Z}^3\}$. The quadratic form $Q_{\delta,\gamma}$, and operator $\Gamma_{\delta,\gamma}$ clearly are $\mathcal{T}$-invariant. According to the standard scheme (see [RS],[K93]), this should lead to some direct integral decomposition, which we are going to describe. Let us consider the following transforms: for $f \in L_2(\mathbf{R}^3)$ we set

$$\widehat{f}(k,x) = \sum_{m \in \mathbf{Z}^3} f(x-m) e^{ik\cdot m}$$

and

$$\widetilde{f}(k,x) = e^{-ik\cdot x} \widehat{f}(k,x), x \in X, k \in K$$

These transforms are correctly defined if we consider the sum as a Fourier series in k variables with values in $L_2(X)$ (see[K93]). We get the isometry $\mathcal{F} : f \to \widetilde{f}$ between $L_2(\mathbf{R}^3)$ and $\int\limits_K^{\oplus} L_2(X)$ (see [RS],[K93]). Since the operator $\Gamma_{\delta,\gamma}$ is $\mathcal{T}$-invariant, it must be decomposable:

$$\Gamma_{\delta,\gamma} = \int\limits_K^{\oplus} \Gamma_{\delta,\gamma}(k) dk,$$

where $\Gamma_{\delta,\gamma}(k)$ is some measurable self-adjoint operator function on K. We are going to describe this decomposition in more detail.

Let us define the form $S^k_{\delta,\gamma}$ on $L_2(X)$ as follows:

$$\mathcal{D}(S^k_{\delta,\gamma}) = \left\{ f \in H^1(X) \, : \, f(x)|_{x_j=1} = e^{ik_j} f(x)|_{x_j=0}, j = 1,2,3 \right\},$$

$$S^k_{\delta,\gamma}[f] = \int\limits_{O_\delta} |\nabla f(x)|^2 dx + \gamma \int\limits_{T_\delta} |\nabla f(x)|^2 dx. \tag{7}$$

It is easy to see that $S^k_{\delta,\gamma}$ is a nonnegative closed quadratic form. Then, a straightforward and easily justified calculation shows that for any $f \in H^1(\mathbf{R}^3)$ we have

$$Q_{\delta,\gamma}[f] = \int_K S^k_{\delta,\gamma}(\widehat{f}(k), \widehat{f}(k))dk,$$

where $\widehat{f}(k)(x) = \widehat{f}(k,x)$. We would also like to get some relation between the quadratic forms in terms of $\widetilde{f}(k)$ instead of $\widehat{f}(k)$. This can be easily done as follows:

$$S^k_{\delta,\gamma}(\widehat{f}(k), \widehat{f}(k)) = \int_{O_\delta} |\nabla \widehat{f}(k,x)|^2 dx + \gamma \int_{T_\delta} |\nabla \widehat{f}(k,x)|^2 dx$$

$$= \int_{O_\delta} |\nabla (e^{ik\cdot x} \widetilde{f}(k,x)|^2 dx + \gamma \int_{T_\delta} |\nabla (e^{ik\cdot x} \widetilde{f}(k,x)|^2 dx$$

$$= \int_{O_\delta} |\nabla_k \widetilde{f}(k,x)|^2 dx + \gamma \int_{T_\delta} |\nabla_k \widetilde{f}(k,x)|^2 dx,$$

where $\nabla_k f(x) = e^{-ik\cdot x}\nabla(e^{ik\cdot x} f(x)) = (\nabla + ik)f(x)$. If we now define the following form in $L_2(\mathbf{T}^3)$:

$$Q^k_{\delta,\gamma}[f] = \int_{O_\delta} |\nabla_k f|^2 dx + \gamma \int_{T_\delta} |\nabla_k f|^2 dx$$

with the domain $H^1(\mathbf{T}^3)$, we get

$$Q_{\delta,\gamma}[f] = \int_K Q^k_{\delta,\gamma}(\widetilde{f}(k), \widetilde{f}(k))dk.$$

Multiplication by $e^{ik\cdot x}$ in $L_2(X)$ provides an isomorphism between $S^k_{\delta,\gamma}$ and $Q^k_{\delta,\gamma}$. In addition, the operator of multiplication by this function depends analytically on $k \in \mathbf{C}^3$, and is an isometry in $L_2(X)$ for $k \in \mathbf{R}^3$. To show measurability of the operator-function $\Gamma_{\delta,\gamma}(k)$, it is sufficient to show it for the operator-function $\Psi(k) = e^{ik\cdot x} \circ \Gamma_{\delta,\gamma}(k) \circ e^{-ik\cdot x}$. In fact, we can prove even more: for any nonreal number $c \in \mathbf{C}$ the operator-function $(\Psi(k) - c)^{-1}$ is analytic with respect to k in a neighborhood of the real space $\mathbf{R}^3$, as a function with values in the space of bounded operators in $L_2(X)$.

Let $\mathcal{E}^2(k) \subset H^2(X)$ be the closed subspace consisting of all functions $f(x) \in H^2(X)$ satisfying the following cyclic conditions:

$$f(x)|_{x_j=1} = e^{ik_j} f(x)|_{x_j=0}, j = 1,2,3 \tag{8}$$

$$\frac{\partial f}{\partial x_j}|_{x_j=1} = e^{ik_j} \frac{\partial f}{\partial x_j}|_{x_j=0}, j = 1,2,3 \tag{9}$$

The space $\mathcal{E}^1(k) \subset H^1(X)$ is defined in an analogous way, with only one condition (8) instead of two. In particular, $\mathcal{D}(S^k_{\delta,\gamma}) = \mathcal{E}^1(k)$.

Due to Theorem 2.2.1 in [K93], $\bigcup\limits_k \mathcal{E}^j(k)(j = 1, 2)$ forms an analytic Hilbert subbundle of the trivial Hilbert bundle $\mathbf{C}^3 \times H^j(X)$. Hence, there is an analytic projector-valued operator function $P_j(k)$ onto this subbundle (see Theorem 1.5.26 in [K93]):

$$P_j(k) : H^j(X) \to \mathcal{E}^j(k), j = 1, 2. \tag{10}$$

LEMMA 3.4. *If $\varphi \in H^1(X)$, and $\varphi \equiv 0$ in a neighborhood of O_δ, then $\varphi \in \mathcal{D}(\Psi(k))$ if and only if $\varphi \in \mathcal{E}^2(k)$. Besides, in this case*

$$\Psi(k)\varphi = -\gamma\Delta\varphi.$$

Proof. Let, first, $\varphi \in \mathcal{E}^2(k)$. Consider the function

$$\psi(x) = e^{-ik\cdot x}\varphi(x).$$

Then, due to the definition of $\mathcal{E}^2(k)$, $\psi \in H^2(\mathbf{T}^3)$. Consider any function $\varphi_1 \in H^1(\mathbf{T}^3) = \mathcal{D}(Q^k_{\delta,\gamma})$. We get

$$S^k_{\delta,\gamma}(\varphi, e^{ik\cdot x}\varphi_1) = Q^k_{\delta,\gamma}(\psi, \varphi_1) = \gamma \int\limits_{T_\delta} \nabla_k\psi \cdot \overline{\nabla_k\phi_1} dx =$$

$$-\gamma \int\limits_{T_\delta} \Delta_k\psi \cdot \overline{\varphi_1} dx = -\gamma \int\limits_{T_\delta} \Delta\varphi \cdot \overline{e^{ik\cdot x}\varphi_1} dx.$$

(We do not have any boundary terms during integration by parts, since φ is supported away from the closure of O_δ.) Since functions $e^{ik\cdot x}\varphi_1(x)$ cover the whole domain of the form $S^k_{\delta,\gamma}$, these equalities show that $\varphi \in \mathcal{D}(\Psi(k))$, and $\Psi(k)\varphi = -\gamma\Delta\varphi$.

Let us assume now that $\varphi \in \mathcal{D}(\Psi(k))$, and $\varphi \equiv 0$ in a neighborhood of O_δ. We will show that $\varphi \in \mathcal{E}^2(k)$. Introducing, as before, $\psi(x) = e^{-ik\cdot x}\varphi(x)$, we conclude that $Q^k_{\delta,\gamma}(\psi, \varphi_1) = (h, \varphi_1)$ for some $h \in L_2(\mathbf{T}^3)$ and all $\varphi_1 \in C^\infty(\mathbf{T}^3)$. This means that ψ is a distributional solution of the elliptic equation $-\gamma\Delta_k\psi = h$ on $\mathbf{T}^3$. Due to ellipticity, we conclude that $\psi \in H^2(\mathbf{T}^3)$, and hence $\varphi \in \mathcal{E}^2(k)$. This finishes the proof of the lemma.□

Let us fix now a function $\rho \in C^\infty(X)$ such that $\rho \equiv 1$ in a neighborhood of ∂X, and $\rho \equiv 0$ in a neighborhood of $\overline{O_\delta}$.

LEMMA 3.5.

(i) A function $\varphi \in H^1(X)$ belongs to $\mathcal{D}(\Psi(k))$, if and only if

1) $\rho\varphi \in \mathcal{E}^2(k)$;
2) $(1 - \rho)\varphi \in \mathcal{D}(\Psi(k))$.

(ii) For functions of the type $f = (1 - \rho)g$ the inclusion $f \in \mathcal{D}(\Psi(k))$ and the value $\Psi(k)f$ do not depend on k, and

$$\Psi(k)f(x) = \begin{cases} -\gamma\Delta f(x) & \text{for} \quad x \in X - \overline{O_\delta}, \\ -\Delta f(x) & \text{for} \quad x \in O_\delta. \end{cases} \tag{11}$$

(iii) The graph norm on $\mathcal{D}(\Psi(k))$ is uniformly with respect to $k \in K$ equivalent to

$$||f|| := (||(1-\rho)f||^2_{\mathcal{D}(\Psi(0))} + ||\rho f||^2_{H^2(X)})^{1/2}. \tag{12}$$

Proof. (i) Let $\varphi \in \mathcal{D}(\Psi(k))$. The same ellipticity arguments, as in the previous lemma show that $\psi(x) = e^{-ik\cdot x}\varphi(x)$ belongs to $H^2_{loc}(\mathbf{T}^3 - \overline{O_\delta})$. Hence, $\rho\psi \in H^2(\mathbf{T}^3)$, and $\rho\varphi \in \mathcal{E}^2(k)$. Due to the previous lemma, $\rho\varphi \in \mathcal{D}(\Psi(k))$, hence $(1-\rho)\varphi \in \mathcal{D}(\Psi(k))$. The converse statement follows from the previous lemma.

(ii) By the definition, $f \in \mathcal{D}(\Psi(k))$ if and only if there exists $h \in L_2(X)$ such that for all $\varphi \in \mathcal{E}^1(k)$ we have

$$\int_{O_\delta} \nabla f \cdot \overline{\nabla\varphi}\, dx + \gamma \int_{R_\delta} \nabla f \cdot \overline{\nabla\varphi}\, dx = \int_X h(x)\overline{\varphi(x)}\, dx \tag{13}$$

If now $f = (1-\rho)g$, then the equality (13) will hold for all $\varphi \in H^1(X)$, and hence it does not depend on k. Choosing functions φ that vanish on G_δ, we get the explicit formulae (11).

(iii) Let us define two norms: the graph norm

$$||f||^2_{\mathcal{D}(\Psi(k))} := ||\Psi(k)f||^2_{L_2(X)} + ||f||^2_{L_2(X)}$$

and the norm (12). We have already shown that for $f \in \mathcal{D}(\Psi(k))$ the function ρf also belongs to $\mathcal{D}(\Psi(k))$, and, due to standard elliptic estimates,

$$||\rho f||_{H^2(X)} \leq C||f||_{\mathcal{D}(\Psi(k))}.$$

In view of the statement (ii) we have

$$||(1-\rho)f||_{\mathcal{D}(\Psi(0))} = ||(1-\rho)f||_{\mathcal{D}(\Psi(k))} \leq C||f||_{\mathcal{D}(\Psi(k))}.$$

This proves the inequality

$$||f|| \leq C||f||_{\mathcal{D}(\Psi(k))}.$$

On other hand

$$||f||_{\mathcal{D}(\Psi(k))} \leq ||(1-\rho)f||_{\mathcal{D}(\Psi(k))} + ||\rho f||_{\mathcal{D}(\Psi(k))}$$

$$\leq ||(1-\rho)f||_{\mathcal{D}(\Psi(0))} + C||\rho f||_{H^2(X)}.$$

This finishes the proof of the lemma.□

Let us introduce now the following functional space:

$$\mathcal{H} = \left\{ f \in H^1(X) \,|\, (1-\rho)f \in \mathcal{D}(\Psi(0)), \rho f \in H^2(X) \right\}$$

with the norm $||f||$ defined by (12). It is easy to show that $\mathcal{H}$ is a Hilbert space.

LEMMA 3.6. *$\mathcal{D}(\Psi(k))$ is a closed subspace in $\mathcal{H}$, and*

$$\bigcup_k \mathcal{D}(\Psi(k))$$

forms an analytic subbundle in $\mathbf{C}^3 \times \mathcal{H}$.

Proof. The closedness follows straightforwardly from the previous lemmas. We can now construct an analytically depending on $k \in \mathbf{C}^3$ projector onto $\mathcal{D}(\Psi(k))$ in $\mathcal{H}$. This is

$$\Pi(k)f = (1-\rho)f + \rho_1 P_2(k)\rho f.$$

Here $\rho_1 \in C^\infty(X)$, $\rho_1 \equiv 1$ on $\mathrm{supp}\{\rho\}$, and $\rho_1 \equiv 0$ in a neighborhood of $\overline{O_\delta}$. The operator $P_2(k)$ is the projector onto $\mathcal{E}^2(k)$ that was introduced in (10). It is rather obvious that $\Pi(k)$ is analytic with respect to k, and projects onto $\mathcal{D}(\Psi(k))$.□

We define on $\mathcal{H}$ the following operator:

$$\Gamma f = \begin{cases} -\gamma\Delta f(x) & \text{for} \quad x \in X - \overline{O_\delta}, \\ -\Delta f(x) & \text{for} \quad x \in O_\delta. \end{cases}$$

It is clear that Γ is bounded as an operator from $\mathcal{H}$ into $L_2(X)$. In addition,

$$\Psi(k) = \Gamma|_{\mathcal{D}(\Psi(k))} \tag{14}$$

Therefore, after trivialization of the bundle $\bigcup\limits_k \mathcal{D}(\Psi(k))$, the operator $\Psi(k)$ becomes an analytic operator function with the values in the space of bounded operators between two Hilbert spaces. Since, for nonreal c the operators $(\Psi(k)-c)^{-1}$ continuously map $L_2(X)$ into $\mathcal{D}(\Psi(k))$, and $\mathcal{D}(\Psi(k))$ is continuously embedded into $L_2(X)$, we conclude that the following statement holds

LEMMA 3.7. *For any non-real $c \in \mathbf{C}$ the operator function $(\Psi(k)-c)^{-1}$ is an analytic (with respect to k in a neighborhood of the real space $\mathbf{R}^3 \subset \mathbf{C}^3$) operator function in $L_2(X)$.*

Now, the direct integral construction of [RS] is applicable, and we can construct a self-adjoint operator

$$\int_K^\oplus \Gamma_{\delta,\gamma}(k)dk.$$

It is a simple exercise to verify that

$$\int_K^\oplus \Gamma_{\delta,\gamma}(k)dk \subset \Gamma_{\delta,\gamma}.$$

Hence, due to self-adjointness, we get the following result

THEOREM 3.8.

$$\int_K^\oplus \Gamma_{\delta,\gamma}(k)dk = \Gamma_{\delta,\gamma}. \tag{15}$$

Now the following statement holds (see [RS],[K93]).

COROLLARY 3.9. *The spectrum of the operator* $\Gamma_{\delta,\gamma}$ *can be represented as*

$$\sigma(\Gamma_{\delta,\gamma}) = \bigcup_{k\in K} \sigma(\Gamma_{\delta,\gamma}(k)).$$

In particular, if for some interval $S=[\alpha,\beta]\subset \mathbf{R}$ *we have* $\sigma(\Gamma_{\delta,\gamma}(k))\cap S=\emptyset$ *for all* $k\in K$*, then* $\sigma(\Gamma_{\delta,\gamma})\cap S=\emptyset$.

4. ONE-DIMENSIONAL CASE. Let $\varepsilon^{(1)}(y), y\in\mathbf{R}$ be the periodic function defined as

$$\varepsilon^{(1)}(y)=\begin{cases}\gamma^{-1} & \text{if} \quad 0\le y<\delta\\ 1 & \text{if} \quad \delta\le y<1\end{cases},\ \varepsilon^{(1)}(y+1)=\varepsilon^{(1)}(y), y\in\mathbf{R}.$$

In other words, $\varepsilon^{(1)}(y)$ is the one-dimensional analogue of the function $\varepsilon(x), x\in\mathbf{R}^3$ (or $\mathbf{R}^2$). We now consider the one-dimensional analogue of the operator Γ that we define as follows: it is the unbounded self-adjoint operator in $L_2(\mathbf{R})$ that corresponds to the quadratic form

$$-\sum_{n=-\infty}^{\infty}\int_{n+\delta}^{n+1}|\nabla f(y)|^2dy-\sum_{-\infty}^{\infty}\gamma\int_{n}^{n+\delta}|\nabla f(y)|^2dy$$

with the domain $H^1(\mathbf{R})$. This operator corresponds to the differential expression

$$\Gamma^{(1)}=-\frac{d}{dy}\frac{1}{\varepsilon^{(1)}(y)}\frac{d}{dy},\ -\infty<y<\infty. \tag{16}$$

It is not hard to show that any function $f(x)$ from the domain of $\Gamma^{(1)}$ belongs to $H^1(\mathbf{R})$, to $H^2[n,n+\delta]$, and to $H^2[n+\delta,n+1]$ with the following conditions on the derivative:

$$\frac{df}{dy_+}|_{n+\delta}=\gamma\frac{df}{dy_-}|_{n+\delta},\ \gamma\frac{df}{dy_+}|_n=\frac{df}{dy_-}|_n, .n\in\mathbf{Z}.$$

It will be convenient to introduce the following parameters:

$$\beta=\gamma\delta^{-2},\ w=\gamma\delta^{-1} \tag{17}$$

Hence,

$$\gamma^{1/2}=w\beta^{-1/2},\ \delta=w\beta^{-1}. \tag{18}$$

THEOREM 4.1. *For every* $C_1>0$ *there exist constants* C_2 *and* C_3 *such that for* $\beta>C_1$ *and* $w<C_2$ *the part* $\sigma(\Gamma^{(1)})\cap[0,\sqrt{C_1}]$ *of the spectrum of the operator* $\Gamma^{(1)}$ *belongs to the union of the intervals*

$$J_n=\{\lambda\ge 0:|\lambda-(\pi n)^2|\le C_3w\}, n\in\mathbf{Z}_+,$$

where $\mathbf{Z}_+$ *is the set of nonnegative integers. In addition to that, each of these intervals* J_n *for* $n \leq \sqrt{C_1}/\pi$ *contains a nonempty portion of the spectrum.*

To prove this theorem, we need the following simple statement of Floquet-Bloch theory [K93],[RS], which can be extracted as a particular case of Theorem 3.8 and of Corollary 3.9.

Proposition 4.2. *Let* $\Gamma^{(1)}(\kappa), 0 \leq \kappa \leq 2\pi$, *be the self-adjoint operator on the interval* $[0,1]$ *defined by the differential operation (16), and by the following boundary conditions:*

$$u(1) = e^{i\kappa}u(0), \ \frac{du}{dy}(1) = e^{i\kappa}\frac{du}{dy}(0). \tag{19}$$

Then the spectrum of the operator $\Gamma^{(1)}(\kappa)$ *is the discrete set of numbers* $\theta_0(\kappa) \leq \theta_1(\kappa) \leq \dots$ *, and the spectrum* $\sigma(\Gamma^{(1)})$ *of the operator* $\Gamma^{(1)}$ *can be described as*

$$\sigma(\Gamma^{(1)}) = \bigcup_{0\leq\kappa\leq 2\pi} \sigma(\Gamma^{(1)}(\kappa)) = \bigcup_{j\geq 0}\bigcup_{0\leq\kappa\leq 2\pi} \{\theta_j(\kappa)\}$$

The spectrum of $\sigma(\Gamma^{(1)}(\kappa))$ (and, correspondingly, of $\sigma(\Gamma^{(1)})$) can be easily found. To do this, one has to solve the following sequence of Cauchy problems:

$$f(0) = a, \ f^{'}(0) = b, \quad -\gamma f^{''}(y) = \lambda f(y), \ \ \text{for} \ \ y \in [0,\delta]$$

$$f(\delta - 0) = f(\delta + 0), \gamma f^{'}(\delta - 0) = f^{'}(\delta + 0), \quad -f^{''}(y) = \lambda f(y), \ \ \text{for} \ \ y \in [\delta, 1]$$

where a, b are arbitrary complex numbers. After this is done, one has to consider the linear mapping in two-dimensional space $T : (a,b) \to (f(1-0), \gamma^{-1}f^{'}(1-0))$. Now, λ is in $\sigma(\Gamma^{(1)}(\kappa))$ if and only if the operator T (which is 2×2 matrix) has the eigenvalue $\exp(ik)$. This straightforward computation leads to the following transcendental equation:

$$2\cos((1-\delta)\mu)\cos(\gamma^{-1/2}\delta\mu) - (\gamma^{1/2} + \gamma^{-1/2})\sin((1-\delta)\mu)\sin(\gamma^{-1/2}\delta\mu) = \tag{20}$$

$$= 2\cos\kappa, \quad \mu = \sqrt{\lambda}.$$

Using the parameters defined in (17) and (18) we can rewrite the equation (20) as follows:

$$(1-\delta)\mu\sin((1-\delta)\mu) = wh(\mu, k), \tag{21}$$

where

$$h(\mu, k) = \tag{22}$$

$$= 2(1-\delta)(1+\gamma)^{-1}\frac{\beta^{-1/2}\mu}{\sin(\beta^{-1/2}\mu)}\left[\cos((1-\delta)\mu)\cos(\beta^{-1/2}\mu) - \cos\kappa\right].$$

LEMMA 4.3. *Let $0 \le \mu \le \beta^{1/2}$. Then the following inequalities hold*

$$|h(\mu,k)| \le 4(1-\delta)\left[(1+\gamma)\sin(1)\right]^{-1} \tag{23}$$

$$|h'_\mu(\mu,k)| \le \tag{24}$$

$$\le (1-\delta)\left(4\beta^{-1/2}\max_{0<\mu<1}\left|\left(\frac{\mu}{\sin\mu}\right)'\right| + 2\left[(1+\gamma)\sin(1)\right]^{-1}(1+\beta^{-1/2})\right)$$

In addition to that, there exists a positive constant C_0 such that for $0 \le \mu < \pi/2$ we have

$$\left| h(\mu,k) - \frac{(1-\delta)}{(1+\gamma)} \times \right. \tag{25}$$

$$\left. \left\{ 2(1-\cos\kappa) + \left[\frac{(1-\cos\kappa)}{3\beta} - ((1-\delta)^2+\beta^{-1})\right]\mu^2 \right\} \right| \le C_0\mu^4,$$

$$\left| h'_\mu(\mu,k) - 2\frac{(1-\delta)}{(1+\gamma)}\left[\frac{(1-\cos\kappa)}{3\beta} - \left((1-\delta)^2+\beta^{-1}\right)\right]\mu \right| \le C_0\mu^3. \tag{26}$$

Proof. The proof of the lemma is simple. The inequalities (23) and (24) are straightforward consequences of (22). The $h(\mu,k)$ as function of μ is obviously smooth and even, hence (25) and (26) follow straightforwardly from the Taylor expansion of the function h defined by (22) at $\mu = 0$.□

Now we provide some estimates for the solutions of the equation (21).

LEMMA 4.4. *Assume that $\beta > C > 0$.*

1) If $w \to 0$, then

$$\gamma \to 0, \delta \to 0; \tag{27}$$

2) There exist positive constants C_1, C_2 such that for $w < C_1$, (21) has solutions $\mu_n \ge 0$ ($n \in \mathbf{Z}_+$, $0 \le n \le \sqrt{C}/\pi$) such that $|\mu_n - \pi n| \le C_2 w$ for $0 < n \le \sqrt{C}/\pi$, and $\mu_0 \le C_2\sqrt{w}$. These points exhaust all roots of (21) on the interval $[0,\sqrt{C}]$.

Remark. the points μ_n depend on k, but all the estimates of the lemma are uniform with respect to k.

Proof. We will consider only nonnegative values of μ. The statement (27) follows immediately from (18). Since we consider only $\mu \in [0,\sqrt{C}]$, we conclude that $\mu < \beta^{1/2}$. Due to (23) and (24), there exists a constant C such that for our values of μ

$$|h(\mu)|, |h'(\mu)| \le C. \tag{28}$$

It follows from Lemma 4.3 and (18) that solutions μ of (21) on the interval $[0, \sqrt{C}]$ must belong to the set

$$S = \{\mu \geq 0 : |(1-\delta)\mu \sin((1-\delta)\mu)| \leq Cw\} \tag{29}$$

or $x = (1-\delta)\mu$ must satisfy the inequality

$$|x \sin x| \leq Cw.$$

This and the inequalities (23), (24) imply for small w that

$$S \subseteq \bigcup_{n \geq 0} S_n, \tag{30}$$

where

$$S_0 = [0, \sqrt{\frac{C'w}{(1-\delta)}}], \;\; S_n = \left\{\mu : |\mu - \frac{\pi n}{1-\delta}| \leq \eta_n = \frac{C'w}{n}\right\}, n \geq 1 \tag{31}$$

for some other constant C'(one has to use Taylor approximation for $x\sin(x)$ and $h(x/(1-\delta))$ at the points $n\pi$). Hence, all solutions belong to the intervals $S_n, n \geq 0$. We note that for $n \leq \sqrt{C}/\pi$ and for small w the intervals S_n belong to $[0, \sqrt{\beta})$. Thus, it remains to prove that in each such interval S_n there is exactly one solution. Consider first the case when $n \geq 1$. In view of (31) and (28), we have, for small w,

$$|(1-\delta)\mu \sin((1-\delta)\mu)||_{\mu = \pi n/(1-\delta) \pm \eta_n} \geq Cn > w \max_{\mu \leq C} |h(\mu)|. \tag{32}$$

This clearly implies that the function $f(\mu) = (1-\delta)\mu\sin((1-\delta)\mu) - h(\mu)w$ alternates its signs on the ends of the interval S_n, and since the function is continuous, it has a zero in S_n. If we consider the derivative $f'(\mu)$ on S_n for small w, than using (28) again we find

$$|f'(\mu)| \geq Cn, \mu \in S_n, \tag{33}$$

which means that the function $f(\mu)$ is monotonic. This fact along with previous results imply that (21) has exactly one solution on each interval $S_n, n \geq 1$. The proof of existence of a unique solution in the interval S_0 can be done in a similar manner using inequalities (25) and (26). The outline of the proof in this case is the following. The function $x \sin x$ behaves around zero as x^2, so a solution of order $\sqrt{w}$ exists for small values of w. Due to (26), the derivative of $wh(\mu)$ behaves as $Cw\mu$, hence, for small w we have $f'(\mu) > 0$ close to the origin. This guarantees the uniqueness of the solution μ_0. The lemma is proven. □

The last lemma clearly implies the following corollary.

Corollary 4.5. *Let $\beta > C > 0$ and w approaches 0. Then the part $\sigma(\Gamma^{(1)}(\kappa)) \cap [0, \sqrt{C}]$ of the spectrum of the operator $\Gamma^{(1)}(\kappa)$ is the discrete set of numbers $\{\theta_0(\kappa), \theta_1(\kappa), \ldots\}$ such that*

$$|\theta_n(\kappa) - (\pi n)^2| \leq C_1 w, \quad n \leq \sqrt{C}/\pi$$

Proof of Theorem 4.1. The statement of theorem is the straightforward consequence of Proposition 4.2 and Corollary 4.5. □

5. AN ESTIMATE FROM ABOVE.. We shall prove in this section an estimate from above for the eigenvalues of the operators $\Gamma_{\delta,\gamma}(k)$ in terms of the eigenvalues of one-dimensional operator $\Gamma^{(1)}(\kappa)$. We consider here the proof of the estimate in three-dimensional case. The proof of analogous estimate in two-dimensional case is literally same. Let us note first that for $x = (x_1, x_2, x_3)$ the following inequality holds:

$$\varepsilon(x) \geq \varepsilon^{(1)}(x_j), j = 1, 2, 3,$$

which immediately implies

$$Q_{\delta,\gamma}(k)[f] = \int_X \varepsilon^{-1}(x)|\nabla f|^2 dx \leq \sum_{1\leq j\leq 3} \int_X \left[\varepsilon^{(1)}(x_j)\right]^{-1} |\partial_j f|^2 dx \tag{34}$$

and therefore

$$\Gamma_{\delta,\gamma}(k) \leq \Gamma^{(1)}(k_1) \otimes I_2 \otimes I_3 + I_1 \otimes \Gamma^{(1)}(k_2) \otimes I_3 + I_1 \otimes I_2 \otimes \Gamma^{(1)}(k_3), \tag{35}$$

$$\text{for } \ k = (k_1, k_2, k_3).$$

Here I_j is the identity operator acting on functions of variable x_j. Let us denote by $\psi_l(k_j)$ and $\theta_l(k_j)$ an orthonormal family of eigenfunctions and the corresponding eigenvalues for the operator $\Gamma^{(1)}(k_j)$. Then obviously the functions $\psi_l(k_1) \otimes \psi_m(k_2) \otimes \psi_n(k_3)$ form an orthonormal basis of eigenfunctions for the operator in the right-hand side of (35). The corresponding eigenvalues are $\theta_l(k_1) + \theta_m(k_2) + \theta_n(k_3)$. This, together with (35) and Corollary 4.5, implies the following statement.

LEMMA 5.1. *Let $\beta > C > 0$ and w approaches 0. Then*

$$\lambda_m(\Gamma_{\delta,\gamma}(k)) \leq \nu_m + C_1 w, m \geq 0 \tag{36}$$

for some $C_1 > 0$.

The last inequality provides the desired estimate from above.

6. SOME AUXILIARY INEQUALITIES. Let us consider the cubes $\Omega_0 = [0,1]^3, \Omega = [0, 1+\delta]^3$, and the following adjacent parallelepipeds defined for $i, j \in \{1, 2, 3\}$:

$$\Omega_j = \{x \, : \, x_j \in [1, 1+\delta], \quad x_i \in [0,1] \ \text{ for } \ i \neq j\},$$

$$\text{for } \ i \neq j : \Omega_{ij} = \{x \, : \, x_i \ \text{ and } \ x_j \in [1, 1+\delta], \quad x_l \in [0,1] \ \text{ for } \ l \neq i, l \neq j\},$$

$$\Omega_{123} = [1, 1+\delta]^3.$$

We also introduce the set

$$\Omega' = \Omega - \Omega_0 = \left(\bigcup \Omega_j\right) \bigcup \left(\bigcup \Omega_{ij}\right) \bigcup \Omega_{123}$$

and the function

$$\gamma(X) = \begin{cases} 1 & \text{if} \quad X \in \Omega_0 \\ \gamma & \text{if} \quad X \in \Omega' \end{cases},$$

where $0 < \gamma < 1$. For a subset $\Xi \subseteq \Omega$ we define

$$\mathcal{B}_\Xi[\varphi] = \int_\Xi \gamma(X)|\nabla\varphi|^2\, dX; \quad ||\varphi||_\Xi^2 = \int_\Xi |\varphi|^2\, dX.$$

We will be interested in some particular pieces of the boundaries of the above domains. Let

$$\Gamma_j = \Omega_j \bigcap \Omega_0; \ \Gamma_{iij} = \Omega_{ij} \bigcap \Omega_i, \Gamma_{ij} = \Omega_{ij} \bigcap \Omega_{123} \ \text{ for } \ i \neq j.$$

Let us consider also some auxiliary objects. We denote by G a parallelepiped in $\mathbf{R}^n$ of the following form:

$$G = [0, a] \times G_1,$$

where G_1 is a parallelepiped in $\mathbf{R}^{n-1}$. In addition to that, we introduce the face

$$\Gamma = \{0\} \times G_1.$$

Lemma 6.1. *If $\varphi \in H^1(G)$, then*

$$||\varphi||_\Gamma^2 \leq 2\left\{a^{-1}||\varphi||_G^2 + a||\nabla\varphi||_G^2\right\}, \tag{37}$$

$$||\varphi||_G^2 \leq 2\left\{a||\varphi||_\Gamma^2 + a^2||\nabla\varphi||_G^2\right\}. \tag{38}$$

Proof. First of all, due to standard embedding theorems, all terms in the inequalities (37), (38) are continuous on the space $H^1(G)$. Hence, it is sufficient to prove the statement of the lemma for a subset of functions that is dense in $H^1(G)$. The space $C^\infty(\overline{G})$ is dense in $H^1(G)$, so we are going to assume that the function φ is smooth up to the boundary. Due to the mean value theorem, it is clear that there exists $x_0 \in [0, a]$ such that

$$\int_{G_1} |\varphi(x_0, y)|^2\, dy \leq a^{-1}||\varphi||_G^2. \tag{39}$$

Then, using the identity

$$\varphi(x, y) = \varphi(x_0, y) + \int_{x_0}^x \varphi_x'(s, y)\, ds, y \in G_1 \tag{40}$$

and Cauchy inequality, we easily obtain

$$|\varphi(x, y)|^2 \leq 2\left\{|\varphi(x_0, y)|^2 + a\int_0^a |\varphi_x'(s, y)|^2\, ds\right\}. \tag{41}$$

Integrating both sides of the last inequality with respect to y over the domain G_1, we get

$$\int_{G_1} |\varphi(x,y)|^2\,dy \le 2\left\{\int_{G_1} |\varphi(x_0,y)|^2\,dy + a\int_{G_1}\int_0^a |\varphi_x'(s,y)|^2\,dsdy\right\}. \tag{42}$$

Setting $x=0$ in the last inequality and using (39) we obtain

$$||\varphi||_\Gamma^2 \le 2\left\{a^{-1}||\varphi||_G^2 + a||\nabla\varphi||_G^2\right\}. \tag{43}$$

This inequality proves (37). Let us turn now to the proof of (38). Consider the inequality (41) for $x_0=0$:

$$|\varphi(x,y)|^2 \le 2\left\{|\varphi(0,y)|^2 + a\int_0^a |\varphi_x'(s,y)|^2\,ds\right\}.$$

Integrating both sides over G we obtain

$$\int_G |\varphi(x,y)|^2\,dxdy \le 2\left\{a||\varphi||_\Gamma^2 + a^2\int_{G_1} dy\int_0^a |\varphi_x'(s,y)|^2\,ds\right\}$$

which gives (38).□

Remark. We would like to point out that in fact the estimates (37),(38) still hold if instead of the whole gradient of the function we put just one of its components. In other words, we actually proved the following inequalities:

$$||\varphi||_\Gamma^2 \le 2\left\{a^{-1}||\varphi||_G^2 + a||\varphi_x'||_G^2\right\}, \tag{44}$$

$$||\varphi||_G^2 \le 2\left\{a||\varphi||_\Gamma^2 + a^2||\varphi_x'||_G^2\right\}. \tag{45}$$

Lemma 6.2. *If $\varphi \in H^1(\Omega)$, then*

$$||\varphi||_{\Omega'}^2 \le A||\varphi||_{\Omega_0}^2 + \max\{A,B\}\mathcal{B}_\Omega[\varphi], \tag{46}$$

where

$$A = 4(11+6\sqrt{2})\delta,\ \ B = (32/3 + 8\sqrt{2} + 2)\delta^2\gamma^{-1}.$$

In addition to that, if $\varphi = 0$ on Ω_0, then

$$||\varphi||_{\Omega'}^2 \le B\cdot\mathcal{B}_\Omega[\varphi]. \tag{47}$$

Proof. Applying (37) to Ω_0 and Γ_j, we get

$$||\varphi||_{\Gamma_j}^2 \le 2\left\{||\varphi||_{\Omega_0}^2 + ||\nabla\varphi||_{\Omega_0}^2\right\} = 2(||\varphi||_{\Omega_0}^2 + \mathcal{B}_{\Omega_0}[\varphi]). \tag{48}$$

Now, using (38) for Ω_j, Γ_j, we obtain

$$||\varphi||_{\Omega_j}^2 \le 2\left\{\delta||\varphi||_{\Gamma_j}^2 + \delta^2||\nabla\varphi||_{\Omega_j}^2\right\}.$$

These two estimates together give

$$||\varphi||^2_{\Omega_j} \le 4\delta(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 2\delta^2||\nabla\varphi||^2_{\Omega_j}. \tag{49}$$

We would like now to estimate $||\varphi||^2_{\Gamma_{iij}}$, considering Γ_{iij} as the base of a parallelepiped that is a part of Ω_i. Namely, let for given i and $j, i \neq j$,

$$\Omega'_{ij} = \{x| : x_i \in [1, 1+\delta],\, x_j \in [1-\delta\sqrt{2}, 1],\, x_l \in [0,1] \text{ for } l \neq i,j\} \subset \Omega_i$$

(we remind the reader that $\delta < 1/2$, as we assumed in the beginning of §2). Applying (37) to Ω'_i and Γ_{iij} , we get the inequality

$$||\varphi||^2_{\Gamma_{iij}} \le 2\left\{\delta^{-1}\sqrt{2}/2||\varphi||^2_{\Omega_i} + \delta\sqrt{2}||\nabla\varphi||^2_{\Omega_i}\right\}$$

Combining with (49), we obtain

$$||\varphi||^2_{\Gamma_{iij}} \le 4\sqrt{2}(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 4\sqrt{2}\delta||\nabla\varphi||^2_{\Omega_i} \tag{50}$$

We will now estimate $||\varphi||^2_{\Omega_{ij}}$. To do this, we apply (38) to Ω_{ij} and Γ_{iij}. This gives:

$$||\varphi||^2_{\Omega_{ij}} \le 2\left\{\delta||\varphi||^2_{\Gamma_{iij}} + \delta^2||\nabla\varphi||^2_{\Omega_{ij}}\right\} \le$$

$$8\delta\sqrt{2}(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 8\delta^2\sqrt{2}||\nabla\varphi||^2_{\Omega_i} + 2\delta^2||\nabla\varphi||^2_{\Omega_{ij}}$$

An analogous estimate holds also, if we use Γ_{jij} and Ω_j:

$$||\varphi||^2_{\Omega_{ij}} \le 8\delta\sqrt{2}(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 8\delta^2\sqrt{2}||\nabla\varphi||^2_{\Omega_j} + 2\delta^2||\nabla\varphi||^2_{\Omega_{ij}}.$$

Summing up the last two inequalities, we get the desired estimate for $||\varphi||^2_{\Omega_{ij}}$:

$$||\varphi||^2_{\Omega_{ij}} \le 8\delta\sqrt{2}(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 4\delta^2\sqrt{2}||\nabla\varphi||^2_{\Omega_i\bigcup\Omega_j} + 2\delta^2||\nabla\varphi||^2_{\Omega_{ij}}. \tag{51}$$

We need now to get an estimate for $||\varphi||^2_{\Gamma_{ij}}$. Let us consider for $i \neq j$ the parallelepiped

$$\Omega''_{ij} = \{x|\, x_i, x_j \in [1, 1+\delta],\, x_l \in [1-\delta\sqrt{2}, 1] \text{ for } l \neq i,j\} \subset \Omega_{ij},$$

and apply to it and its face Γ_{ij} the estimate (37) combined with (51). This gives the following inequality

$$||\varphi||^2_{\Gamma_{ij}} \le 2\left\{\delta^{-1}\sqrt{2}/2||\varphi||^2_{\Omega_{ij}} + \delta\sqrt{2}||\nabla\varphi||^2_{\Omega_{ij}}\right\} \le$$

$$16(||\varphi||^2_{\Omega_0} + \mathcal{B}_{\Omega_0}[\varphi]) + 8\delta||\nabla\varphi||^2_{\Omega_i\bigcup\Omega_j} + 4\sqrt{2}\delta||\nabla\varphi||^2_{\Omega_{ij}} \tag{52}$$

Then we use (38) for Ω_{123} and its face Γ_{ij} , which leads the estimate

$$||\varphi||^2_{\Omega_{123}} \le 2\left\{\delta||\varphi||^2_{\Gamma_{ij}} + \delta^2||\nabla\varphi||^2_{\Omega_{123}}\right\} \le$$

$$(53)\quad 32\delta(||\varphi||^2_{\Omega_0}+\mathcal{B}_{\Omega_0}[\varphi])+16\delta^2||\nabla\varphi||^2_{\Omega_i\bigcup\Omega_j}+8\sqrt{2}\delta^2||\nabla\varphi||^2_{\Omega_{ij}}+2\delta^2||\nabla\varphi||^2_{\Omega_{123}}$$

We can get estimates similar to (53) for any of three faces Γ_{ij}. Averaging these three estimates, we come up with the following inequality:

$$(54)\qquad ||\varphi||^2_{\Omega_{123}}\le 32\delta(||\varphi||^2_{\Omega_0}+\mathcal{B}_{\Omega_0}[\varphi])+(32/3)\delta^2||\nabla\varphi||^2_{\bigcup\Omega_j}+$$

$$(8\sqrt{2}/3)\delta^2||\nabla\varphi||^2_{\bigcup\Omega_{ij}}+2\delta^2||\nabla\varphi||^2_{\Omega_{123}}.$$

Now, in order to estimate $||\varphi||^2_{\Omega'}$, we have to add the estimates (49),(51), and (54). This leads to the following inequality:

$$||\varphi||^2_{\Omega'}\le 4(11+6\sqrt{2})\delta(||\varphi||^2_{\Omega_0}+\mathcal{B}_{\Omega_0}[\varphi])+(32/3+8\sqrt{2}+2)\delta^2||\nabla\varphi||^2_{\bigcup\Omega_j}+$$

$$(8\sqrt{2}/3+2)\delta^2||\nabla\varphi||^2_{\bigcup\Omega_{ij}}+2\delta^2||\nabla\varphi||^2_{\Omega_{123}}.$$

This, in turn, implies

$$||\varphi||^2_{\Omega'}\le 4(11+6\sqrt{2})\delta(||\varphi||^2_{\Omega_0}+\mathcal{B}_{\Omega_0}[\varphi])+(32/3+8\sqrt{2}+2)\delta^2||\nabla\varphi||^2_{\Omega'}.$$

Thus, we can conclude that

$$||\varphi||^2_{\Omega'}\le A||\varphi||^2_{\Omega_0}+\max\{A,B\}\mathcal{B}_{\Omega}[\varphi],$$

where

$$A=4(11+6\sqrt{2})\delta,\ \ B=(32/3+8\sqrt{2}+2)\delta^2\gamma^{-1}.$$

This proves (46). The validity of (47) is also clear from the previous calculations.□

We will usually consider functions in the unit cube X, where the cube O_δ plays the role of Ω_0, and R_δ plays the role of Ω' (see the notation in the beginning of the paper). Since this new picture can be obtained from the one considered in this section by the contraction with the factor $(1-\delta)/(1+\delta)$, an obvious recalculation in (46) and (47) leads to the following statement

Corollary 6.3. *If $\varphi\in H^1(X)$, then*

$$(55)\qquad ||\varphi||^2_{R_\delta}\le A||\varphi||^2_{O_\delta}+(1-\delta)^2(1+\delta)^{-2}\max\{A,B\}\mathcal{B}_X[\varphi].$$

In addition, if $\varphi=0$ on O_δ, then

$$(56)\qquad ||\varphi||^2_{R_\delta}\le\max\{A,B\}(1-\delta)^2(1+\delta)^{-2}\mathcal{B}_X[\varphi].$$

Remark. Some estimates of the same type (with better values of constants) can be easily obtained for the two-dimensional case. The proof is exactly the same, only we have to deal with three adjacent rectangles instead of seven parallelepipeds.

7. AN ESTIMATE FROM BELOW. Let us denote by $\nu'_m = (1-\delta)^{-2}\nu_m$ the eigenvalues of the Neumann Laplacian in O_δ, counted in increasing order with their multiplicity (we remind the reader that ν_m are defined by (3)). We are ready now to prove our estimate from below for the spectrum of the operator $\Gamma_{\delta,\gamma}(k), k \in K$ (see the notation in the beginning of the paper). We denote by $\lambda_j = \lambda_j(\Gamma_{\delta,\gamma}(k))$ the eigenvalues of the operator $\Gamma_{\delta,\gamma}(k)$, counted in increasing order with their multiplicity.

THEOREM 7.1. *Let N be a positive number, and ν'_m be the largest eigenvalue of the Neumann Laplacian in O_δ that is smaller than N. Then, for sufficiently small values of δ and $\gamma^{-1}\delta^2$ (it is sufficient, for instance, that $(32/3+8\sqrt{2}+2)\delta^2\gamma^{-1}(1-\delta)^2(1+\delta)^{-2}N < 1$) the following inequality holds for all $k \in K$:*

$$\lambda_j(\Gamma_{\delta,\gamma}(k)) \geq (1-\alpha_N)\nu_j, \quad j = 1, ..., m, \tag{57}$$

where

$$\alpha_N = 4(11+6\sqrt{2})\delta +$$

$$\max\{4(11+6\sqrt{2})\delta, (32/3+8\sqrt{2}+2)\delta^2\gamma^{-1}\}(1-\delta)^2(1+\delta)^{-2}N. \tag{58}$$

Proof. Using the minimax principle, we can represent the eigenvalues as follows:

$$\lambda_j(\Gamma_{\delta,\gamma}(k)) = \min_{V^j} \max_{||\varphi||^2_X=1, \varphi\in V^j} S^k_{\delta,\gamma}[\varphi], \tag{59}$$

where V^j runs over the set of all j-dimensional subspaces of the space $\mathcal{D}(S^k_{\delta,\gamma}) \subset H^1(X)$. First of all, if $\lambda_j \geq N$, then there is nothing to prove, since $\lambda_j \geq N > \nu_j$. So, we may assume that $\lambda_j < N$. Let us pick an arbitrary positive τ such that $\lambda_j < N - \tau$. In view of (59), we can find a j-dimensional space V^j such that

$$\lambda_j \geq -\tau + \max_{||\varphi||^2_X=1, \varphi\in V^j} S^k_{\delta,\gamma}[\varphi]. \tag{60}$$

In particular, this inequality and the choice of τ clearly give

$$\max_{||\varphi||^2_X=1, \varphi\in V^j} S^k_{\delta,\gamma}[\varphi] < N. \tag{61}$$

This, along with Corollary 6.3, implies the following inclusion:

$$\{\varphi : ||\varphi||^2_X = 1, \varphi \in V^j\} \subseteq \{\varphi : 1 \geq ||\varphi||^2_{O_\delta} \geq 1-\alpha_N, \varphi \in V^j\}. \tag{62}$$

In view of (7) $S^k_{\delta,\gamma}[\varphi] \geq \mathcal{B}_{O_\delta}[\varphi] = \int_{O_\delta} |\nabla\varphi|^2\, dX$, so the inequality (60) implies:

$$\lambda_j \geq -\tau + \max_{||\varphi||^2_X=1, \varphi\in V^j} \mathcal{B}_{O_\delta}[\varphi]. \tag{63}$$

Since on V^j, due to (63) we have the estimate $\mathcal{B}_{O_\delta}[\varphi] \leq N||\varphi||^2_X$, the inequality (55) implies that

$$\lambda_j \geq -\tau + \max_{||\varphi||^2_{O_\delta}=1-\alpha_N, \varphi\in V^j_\tau} \mathcal{B}_{O_\delta}[\varphi], \tag{64}$$

where V^j_τ is the linear space of the restrictions of functions $\varphi \in V^j$ onto the set O_δ. In view of the inequality (56), it is easy to see that the restriction mapping is one-to-one for sufficiently small values of δ and $\gamma^{-1}\delta^2$ (for instance, for $(32/3 + 8\sqrt{2} + 2)\delta^2\gamma^{-1}(1 - 2\delta)^2 N < 1$). Hence, the space V^j_τ is also j-dimensional. The inequality (64) clearly implies

$$\lambda_j \geq -\tau + \min_{V^j(O_\delta)} \max_{||\varphi||^2_{O_\delta} = 1-\alpha_N, \varphi \in V^j(O_\delta)} \mathcal{B}_{O_\delta}[\varphi] \tag{65}$$

$$= -\tau + (1 - \alpha_N) \min_{V^j(O_\delta)} \max_{||\varphi||^2_{O_\delta} = 1, \varphi \in V^j(O_\delta)} \mathcal{B}_{O_\delta}[\varphi], \tag{66}$$

where $V^j(O_\delta)$ runs over the set of j-dimensional subspaces of the Sobolev space $H^1(O_\delta)$. The last inequality can be rewritten as

$$\lambda_j \geq -\tau + (1 - \alpha_N)\nu'_j.$$

Since τ is arbitrary, this implies the desired inequality (57). The lemma is proven.□

8. PROOF OF THE MAIN RESULT. We are ready now to prove the main Theorem 2.1. In fact, the proof consists in putting together several statements that have been proven already. Simple combination of the Corollary 3.9, Lemma 5.1, and Theorem 7.1 results in (4).□

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