THEORY OF DISORDERED SPIN SYSTEMS

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Disordered, i.e., containing random parameters, lattice spin systems are considered. It is shown that the free energy in the macroscopic limit becomes nonrandom if the probability distribution of the random parameters satisfies conditions of spatial homogeneity on the average and vanishing of statistical correlations at distant points. The possible orientations of the spins in these systems are discussed in terms of random fields. An asymptotically exactly solvable model of such a system is proposed; it demonstrates different types of orientation, including one corresponding to the spin glass state in which there is no macroscopic magnetization but the magnetic moment of individual regions of the crystal is nonzero.

1. Introduction

Dilute solutions of atoms of transition metals with large magnetic moment (Fe, Co, Mn) in paramagnets (Cu, Au) have for long attracted the interest of experimentalists and theoreticiams (see [1-4] and the literature quoted there). These systems have a number of rather unusual properties, among which we mention the very sharp peak in the graph $\chi(T)$ of the magnetic susceptibility in zero field and the linear dependence in the limit $T \to 0$ of the specific heat on T with a slope independent of the impurity concentration. It was recognized comparatively long ago [2] that these and many other properties of these solutions are due to the indirect Ruderman–Kittel–Kasuya–Yosida interaction of the spins of the impurity atoms brought about by the exchange of matrix electrons; it has the form

$$J(|x-y|)\mathbf{s}_x\mathbf{s}_y, \quad J(r)=(k_F r)^{-3}\cos 2k_F r, \tag{1.1}$$

where $\mathbf{s}_x$ are the spins of the impurities and k_F is the Fermi momentum. The rapid oscillations and weak decrease of $J(r)$ (1.1), and also the random distribution of the impurities lead at sufficiently low temperatures to a "freezing" of the spins in random directions and to an increase in χ. The resulting magnetic structure is called a spin glass. It is a "conglomerate" of blocks of only slightly disoriented "frozen" spins whose total orientation however varies from block to block, so that the macroscopic moment is zero.

In [3], it was suggested that this freezing of the spins should be regarded as a phase transition. It is not however clear how one can solve the statistical problem corresponding to the interaction (1.1), i.e, a Hamiltonian of the form

$$H=\frac{1}{2}\sum_{x\neq y} J(|x-y|)\mathbf{s}_x\mathbf{s}_y c_x c_y - \sum_x \mathbf{h}\mathbf{s}_x c_x, \tag{1.2}$$

where c_x is the "population number" of site x, equal to 1 with probability c and 0 with probability $1 - c$, where c is the concentration of the impurity atoms. The difficulty of solving this problem is also increased by the fact that because the considered system is disordered* it is necessary to average over the positions of the impurity atoms, i.e., over the values of $\{c_x\}$, the free energy and not the partition function (the case of "quenched" impurities, i.e., impurities that are not in equilibrium with the matrix). In [3], it was therefore suggested that (1.2) should be replaced by the expression

$$\frac{1}{2}\sum_{x\neq y} J_{xy}\mathbf{s}_x\mathbf{s}_y - \sum_x \mathbf{h}\mathbf{s}_x, \tag{1.3}$$

* To avoid misunderstanding, note that when here and below we speak of disordered (respectively, ordered) systems we mean systems that have (or do not have) in them random parameters (the positions of impurities, etc), and the terminology should not be confused with the expression disordered (respectively, ordered) state frequently used in statistical physics to designate the state of matter above (below) the phase transition point.

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in which J_{xy} are independent random variables with symmetric Gaussian distribution and $\langle J_{xy}^2 \rangle = J^2$. In the opinion of Edwards and Anderson [3] this would preserve the basic feature of (1.2) — the strongly oscillating nature of $J(r)$ in (1.1).

Using the self-consistent field approximation, Edwards and Anderson [3] found in this model a point of inflection in the temperature dependence of χ, and they interpreted this as evidence for the existence of a phase transition to the spin glass state. Later, in [4] an attempt was made to give the calculations in [3] an asymptotically exact meaning, and for this it was assumed that

$$J_{xy}=-N^{-1}\tilde{J}_0+N^{-1/2}\tilde{J}_{xy}, \tag{1.4}$$

where $\tilde{J}_0$ is a positive constant, and $\tilde{J}_{xy}$ are random variables as in (1.3). It is natural to expect such a model in the limit $N \to \infty$ to give an exact answer agreeing with the one obtained in the average field approximation since such a situation obtains in the case of an ordered ferromagnet [5] when $\tilde{J}_{xy} = 0$. According to [4], this really is so — the result obtained in [4] agrees with the one found in [3]. However, the calculations made in [4] include some assumptions that are difficult to verify and are apparently not completely correct, which has the consequence that the entropy obtained in [4] is negative at low temperatures, as the authors themselves note. But this, as will be shown below on the basis of very simple general considerations, is impossible in the case of the Ising model considered in [4]. Note also that a spherical model with the interaction (1.4) was considered in [6], which also contains a discussion of [4].

The present paper consists of two parts. In Sec. 2, we study the general properties of disordered spin systems, both classical and quantum. We show that if the random interaction J_{xy} has the properties of spatial homogeneity and the statistical correlations between its values at pairs of distant points tend to zero, then the free energy of such a system in the macroscopic limit is nonrandom, i.e., it is in practice a certain quantity. Under the specified conditions, this justifies the usually employed procedure for calculating the mean value with respect to the random parameters of the free energy and the identification of it with the observable free energy. In the same section, we prove that the entropy in the Ising model and in quantum systems is positive at all temperatures. In Sec. 3 (in its first part) we give an heuristic description of the possible states of disordered spin systems. The main part of this section is devoted to considering models (classical and quantum) of such systems that differ from (1.4). Namely, we assume that the interaction in these models has the "separable" form

$$J_{xy}=-N^{-1}\sum_{k=1}^{n_1} f_k\alpha_x^{(k)}\alpha_y^{(k)}+N^{-1}\sum_{k=1}^{n_2} a_k\alpha_x^{(k+n_1)}\alpha_y^{(k+n_1)}, \tag{1.5}$$

where f_k and a_k are positive constants, and $\alpha_x^{(k)}$ are random and, in general, dependent variables, satisfying for each k the conditions of spatial homogeneity and vanishing of correlations as $|x - y| \to \infty$. The presence of the factor N^{-1} suggests that, as in (1.4), this model in the limit $N \to \infty$ must give results that coincide with the ones obtained in the self-consistent field approximation. It can be shown that this is really so, and, in contrast to (1.4), this can be proved rigorously. The arguments employed here are a generalization of the method developed in [7] for investigating model Hamiltonians that admit an asymptotically exact solution in the macroscopic limit.

Our results, in particular the point of inflection in the graph of $\chi(T)$ at a certain temperature T_c (see (3.29)) are valid in the general case of statistically dependent and arbitrarily distributed $\alpha_x^{(k)}$ and to a considerable extent do not depend on the form of this distribution. However, it seems to us that the following form of the probability density $\alpha_x^{(k)}$ is of particular interest:

$$p(\alpha)=(1-c)\delta(\alpha)+cq(\alpha), \tag{1.6}$$

where $0\leqslant c\leqslant 1$, $q(\alpha)\geqslant 0$, $\int q(\alpha)\,d\alpha=1$. In accordance with (1.2), this form of $p(\alpha)$ corresponds to the case when an impurity atom is present or absent at every point of the lattice with probability c or 1 − c. From the mathematical point of view, this assumption means that the random variables $\alpha_x^{(k)}$ in (1.5) are replaced by $\gamma_x^{(k)}c_x$, where the two spatially homogeneous sequences $\gamma_x^{(k)}$ and c_x are assumed to be independent of one another, which, of course, does not rule out the possibility of a statistical dependence between the terms of each sequence (it is only necessary that the statistical correlations between these terms tend to zero as $|x - y| \to \infty$, the rate at which this happens being unimportant for what follows). Thus, in our model $\gamma_x^{(k)}$ simulates the oscillations of the exchange integrals (1.1), and c_x determine the configuration of the impurity atoms in the sample. By means of these quantities, we introduce into the theory a dependence on the impurity concentration, which was not considered in [3, 4]. It can then be shown that the critical temperature at which the graph of $\chi(T)$ has a point of inflection is proportional to the concentration (formula (3.18b)), and in the Ising model and in the quantum models the low-temperature specific heat has the observed [2] linear dependence with slope that is independent of the concentration (Eq. (3.33)).

2. General Properties of Disordered Spin Systems

1. For simplicity, we shall consider the classical isotropic Heisenberg model [5], although all the results obtained below also remain true for general classical and quantum spin models.

Thus, in d-dimensional ($d = 1, 2, 3$) space we consider a simple lattice with unit cell formed by the basis vectors $\mathbf{a}_1, \ldots, \mathbf{a}_d$, and we take V to be a parallelepiped with sides of length $N_1 a_1, \ldots, N_d a_d$, $a_j = |\mathbf{a}_j|$. For every point x of the lattice and every pair of points (x, y) we specify random variables $\mathbf{h}_x(\omega)$ and $J_{xy}(\omega)$, which are an external field and the exchange integrals (here, ω is a point of the space Ω of the possible realizations of these random variables). We shall assume that $\mathbf{h}_x$ and J_{xy} have the properties of spatial homogeneity and vanishing of the correlations at distant points. The first of them expresses translational invariance on the average, which holds in a disordered system of macroscopic dimensions, and can be formulated as the condition that all mean values* are independent of a:

$$\langle \mathbf{h}_{x_1+a} \ldots \mathbf{h}_{x_{k_1}+a} J_{y_1+a, z_1+a} \ldots J_{y_{k_2}+a, z_{k_2}+a} \ldots \mathbf{h}_{t_1+a} \ldots \rangle ,$$

and the second property follows from the requirement that sufficiently distant parts of the system should have a small influence, in the statistical sense, on one another and is expressed by the decomposition as $a \to \infty$ of mean values of the form

$$\langle \mathbf{h}_{x_1} \ldots \mathbf{h}_{x_{k_1}} J_{y_1 z_1} \ldots J_{y_{k_2}, z_{k_2}} \ldots \mathbf{h}_{u_1+a} \ldots \mathbf{h}_{u_{l_1}+a} J_{v_1+a, w_1+a} \ldots J_{v_{l_2}+a, w_{l_2}+a} \rangle$$

into a product of the corresponding mean values.

It is readily seen that the random variables that occur in (1.2), i.e., $\mathbf{h} = \text{const}$ and J_{xy} of the form $J(|x - y|)c_x c_y$, where $J(|x|)$ is a nonrandom function and c_x are the population numbers, satisfy these conditions if the random variables c_x satisfy them. The simplest but important example of such c_x is provided by statistically independent and identically distributed random variables, which is a sensible choice for c_x at low impurity concentrations.

The properties we have assumed for $\mathbf{h}_x$ and J_{xy} can be conveniently formalized by introducing, as is usually done in ergodic theory [8], a shift operator T_a, which acts on the space of realizations Ω is such a way that

$$\mathbf{h}_x(T_a \omega) = \mathbf{h}_{x+a}(\omega), \tag{2.1}$$

$$J_{x,y}(T_a \omega) = J_{x+a, y+a}(\omega). \tag{2.2}$$

Then spatial homogeneity is equivalent [8] to the operator T_a preserving the probabilities of all events, and the ergodic theorem is true: for any function $F(\omega)$ on Ω there exists the limit

$$\lim_{N\to\infty} N^{-1} \sum_{a\in V} F_a(T_a \omega) = \bar{F}(\omega) \tag{2.3}$$

($N = N_1 N_2 \ldots N_d$), and the property of correlation weakening means that this limit is a determinate quantity, equal to the limit of the mean values of $F(\omega)$, i.e.,

$$\bar{F} = \langle F \rangle. \tag{2.4}$$

Note that the very fact of the existence of the limit (2.3) is also true in the case when the summation is performed over only a certain sublattice since invariance under shifts over the sublattice follows from invariance under all shifts. We shall make use of this remark in what follows.

2. Having $\mathbf{h}_x$ and J_{xy}, we can write down the Hamiltonian of the classical Heisenberg model:

$$H_N = -\frac{1}{2} \sum_{x\neq y,\, x,y\in V} J_{xy} \mathbf{s}_x \mathbf{s}_y - \sum_{x\in V} \mathbf{h}_x \mathbf{s}_x, \tag{2.5}$$

* The angular brackets $\langle \ldots \rangle$ here and below denote averaging over the realization of the random variables contained within them.

where $\mathbf{s}_x$ is the spin at the point x, which we shall assume is a D-dimensional unit vector (in the case D = 1, $\mathbf{s}_x$ takes the values ±1, which corresponds to the Ising model). The free energy f_N corresponding to (2.5) is defined in the usual manner as

$$f_N = -(\beta N)^{-1} \ln Z_N, \quad Z_N = \int e^{-\beta H_N} \prod_{x \in V} ds_x, \tag{2.6}$$

where β^{-1} is the temperature, and ds_x is the surface element of the (D − 1)-dimensional sphere of unit area (for D = 1, we have $2^{-N} \sum_{\{s_x, x \in V\}} \ldots$) in (2.6).

THEOREM. Suppose

$$\sum_x \langle |J_{0x}| \rangle < \infty, \quad \langle |h_0| \rangle < \infty. \tag{2.7}$$

Then in the limit $V \to \infty$* the free energy f_N in (2.6) tends with probability 1 to a nonrandom limit f, and

$$f = \lim_{V \to \infty} \langle f_N \rangle. \tag{2.8}$$

The proof of the theorem has much in common with that of the existence of thermodynamic limits in ordered systems, and we therefore give only the outline. The proof has two main parts. The first is to obtain an upper bound for the free energy that is uniform in V. Since $|\mathbf{s}_x| = 1$, it follows from (2.5) and (2.6) that

$$|f_N| \leqslant (2N)^{-1} \sum_{x \in V} J_x + N^{-1} \sum_{x \in V} |h_x|, \tag{2.9}$$

where $J_x = \sum_y |J_{xy}|$ by virtue of (2.2) and (2.7) is a metrically transitive field [8]. Equation (2.9) is the required estimate since in accordance with (2.3), (2.4), and (2.7) the right-hand side of (2.9) has a finite limit as $V \to \infty$ with probability 1.

The second part is the splitting of the original lattice, and hence the region V, into congruent parallelepipeds V_i separated by "corridors" of width R. These last can be conveniently regarded as the union of layers $\tilde{V}_i$ of thickness R/2 surrounding each parallelepiped V_i, and one can set $\tilde{V} = \bigcup_i \tilde{V}_i$. Then it can be shown that

$$\left| f_N - b_V n_V^{-1} \sum_i f_{N_i} \right| \leqslant N^{-1} \sum_{x \in V} J_R(x) + N^{-1} \sum_{x \in \tilde{V}} (2J_x + h_x), \tag{2.10}$$

where n_V is the number of the V_i lying entirely in V, $b_V = n_V N^{-1} \to |V_i \cup \tilde{V}_i| = b$, $J_R(x) = \sum_{|y|>R} |J_{0y}|$ by virtue of (2.2) and (2.7) is a metrically transitive field. In accordance with (2.3), the arithmetic mean on the left-hand side of (2.10) has a finite limit with probability 1. The same is true for both terms on the right-hand side since the second of them can be rewritten in the form of the arithmetic mean of the $I_i = \sum_{x \in \tilde{V}_i} (2J_x + |h_x|)$.

Therefore

$$\overline{\lim_{V \to \infty}} f_N - \underline{\lim_{V \to \infty}} f_N \leqslant 2 \sum_{|y| \geqslant R} \langle |J_{0y}| \rangle + 2b\xi_I, \tag{2.11}$$

$$\xi_I = \langle I_i | \mathfrak{A} \rangle, \tag{2.12}$$

where $\langle \ldots | \mathfrak{A} \rangle$ denotes the conditional mean value with respect to the σ algebra $\mathfrak{A}$ of sets that are invariant under shifts over the sublattice composed of the centers of the parallelepipeds V_i [8]. But by virtue of (2.7), $\langle \xi_I \rangle = |\tilde{V}_i| \left(\sum_y \langle |J_{0y}| \rangle + \langle |h_0| \rangle \right) < \infty$. Therefore, choosing V_i and $\tilde{V}_i$ such that $|V_i|, |\tilde{V}_i| \to \infty$, $|\tilde{V}_i| \cdot |V_i|^{-1} \to 0$, we

* As was noted above, V is a parallelepiped and then $V \to \infty$ means that the lengths of all its sides $N_1, \ldots, N_d \to \infty$. By means of arguments similar to those used in [9], one can extend the assertion of the theorem to a larger class of regions that tend to infinity in van Hove's sense.

can arrange that $\langle \xi_I \rangle \to 0$. Since $\xi_I \geq 0$, there exists a subsequence V_i, and $\tilde{V}_i$, for which ξ_I tends to zero with probability 1. Since the left-hand side of (2.11) does not depend on the choice of V_i and $\tilde{V}_i$, what we have said means that with probability 1 there exists a limit f of the sequence of free energies f_N as $V \to \infty$:

$$\lim_{V \to \infty} f_N = f. \tag{2.13}$$

The above arguments do not yet prove that the limit f is not a random variable. But it is easy to see that $f(T_a\omega) = f(\omega)$ for all a and ω, and then f cannot depend on ω since otherwise we would arrive at a contradiction with the assumption that the correlations vanish at infinitely distant points [8]. With regard to the relation (2.8), it is a consequence of (2.9) and can be established by means of standard arguments in ergodic theory.

We shall give the name self-averaging to the property of the free energy just proved. It is natural to expect that not only the free energy but also the other quantities that characterize the disordered system (magnetization, susceptibility) have a similar property, i.e., they become certain in the macroscopic limit. But, in contrast to the free energy, the question of the self-averaging of physical quantities cannot be resolved in the affirmative in the general case on account of the possible existence of phase transitions, at which certain quantities cease to be definite. For such quantities, which can be represented in the form of the first derivatives of the free energy with respect to a parameter that occurs linearly in the Hamiltonian (magnetization, mean energy) and such that the phase transitions with respect to this parameter can be studied an affirmative answer at parameter values not lying on the transition lines follows from the theorem proved above and Griffith's theorem [10], which asserts that the derivatives of a convergent sequence of convex functions have as limit the derivative of the limit function at the points where this derivative is continuous. With regard to the derivatives of higher order (susceptibility, specific heat), even in the models considered below the proof of their self-averaging requires very detailed estimates whose derivation would go beyond the scope of the present paper, whereas the verification of this fact in the case of the first derivatives can also be achieved without recourse to Griffith's general theorem by means of the method we used to calculate the free energy. We shall not enter into a further discussion of this question but merely point out that it is a generalization of the question relating to the existence of macroscopic limits of the correlation functions (states) in ordered systems, and it is therefore clear why it is complex.

3. We define the entropy of the pre-limit (in finite volume) spin system by, as usual, the relation

$$S_N = -\partial f_N / \partial T. \tag{2.14}$$

By direct differentiation, one can show that $\partial^2 \ln Z_N / \partial \beta^2 \geqslant 0$, $\beta = 1/kT$, i.e., $\ln Z_n$ is a convex function of β. But then $-f_N = (\beta N)^{-1} \ln Z_N$ is a convex function of β^{-1}, i.e., the temperature, and this means in accordance with (2.14) that

$$\partial S_N / \partial T \geqslant 0. \tag{2.15}$$

Therefore, there exists the limit $S_N^{(0)} = \lim_{T \to 0} S_N$. In the quantum case and in the Ising model, when $Z_N = \mathrm{Sp} \exp(-\beta H_N)$, $S_V^{(0)} = N^{-1} k \ln \varkappa$, where $\varkappa$ is the multiplicity of the ground-state degeneracy (i.e., $\varkappa \geq 1$). Therefore $S_V^{(0)} \geqslant 0$, and then by virtue of (2.15)

$$S_N \geqslant 0 \tag{2.16}$$

at all temperatures. By what we have proved above, f_N is a concave function of the temperature ($\partial^2 f_N / \partial T^2 \leqslant 0$) and tends with probability 1 to f as $N \to \infty$, so that in accordance with Griffith's theorem [10] there exists at all T (except perhaps for a countable set of values) and with probability 1 a nonrandom limit

$$S = \lim_{V \to \infty} S_N,$$

for which the inequalities (2.15) and (2.16) are therefore also satisfied. Thus, for quantum systems and for the Ising model the entropy of the limiting infinite system everywhere where it is defined is a non-negative and nondecreasing function of the temperature. We mention here this simple and, essentially, well-known fact because the entropy obtained in [4], as is noted by the authors themselves, is negative at low temperatures.

In this connection, we note the following. Using the inequality between the geometric and arithmetic means, we can write $\langle \ln Z_N \rangle = \varepsilon^{-1} \langle \ln Z_N^{\varepsilon} \rangle \leqslant \varepsilon^{-1} \ln \langle Z_N^{\varepsilon} \rangle$ for $\varepsilon > 0$. Hence, by virtue of the proved theorem,

$$f \leqslant \varepsilon^{-1} \lim_{N \to \infty} N^{-1} \ln \langle Z_N^{\varepsilon} \rangle.$$

The right-hand side of this inequality was calculated in [4] under certain assumptions for integral ε and after "continuation" to the value $\varepsilon = 0$ was regarded in [4] as the exact expression for the free energy on the basis of the fact that $\lim_{\varepsilon\to 0} \varepsilon^{-1}(x^\varepsilon - 1) = \ln x$.

3. Exactly Solvable Models of Spin Glass

1. Before we consider our models, we consider one interpretation of the orientations of the moments as observed in disordered spin systems. This interpretation is essentially already contained in [3], but neither there nor subsequently was it sufficiently clearly presented. Nevertheless, it is very perspicuous, although we shall not argue everywhere with rigor in its exposition.

We note that in the considered systems there are two procedures for averaging – over the Gibbs distribution and over the realizations of the random variables $\mathbf{h}_x$ and J_{xy}. We shall denote these two procedures by $\langle \ldots \rangle_G$ and $\langle \ldots \rangle$. Then $\mathbf{m}_x = \langle \mathbf{s}_x \rangle_G$, which does not depend on x in an infinite ordered system, does in a disordered case, and the dependence is different in different realizations. In other words, $\mathbf{m}_x$ in the disordered case is a random field on the lattice. Important properties of this field are its spatial homogeneity and the vanishing of the correlations between the values at distant points in agreement with the picture described in the introduction and confirmed below for the distribution of the spins, this consisting of blocks with strong statistical correlations of the directions of the moments within them and weak correlation between them. Therefore, the macroscopic magnetization of the system **M**, defined as

$$\mathbf{M} = \lim_{N\to\infty} N^{-1} \sum_{x\in V} \langle \mathbf{s}_x \rangle_G \, c_x,$$

must by virtue of the ergodic theorem (2.3)-(2.4) be equal to

$$\mathbf{M} = \langle \mathbf{m}_x \rangle \equiv \langle c_x \langle \mathbf{s}_x \rangle_G \rangle. \tag{3.1}$$

In the absence of an external magnetic field and at sufficiently high temperature, the field $\mathbf{m}_x$ is identically zero, which corresponds to the paramagnetic state of matter. But with decreasing temperature, one may also obtain a state in which the field $\mathbf{m}_x$ is nonzero. If at the same time $\langle \mathbf{m}_x \rangle = \mathbf{m} \neq 0$, then it must be regarded as the disordered analog of a ferromagnet since $\mathbf{M} = \mathbf{m} \neq 0$ by virtue of (3.1). But if $\mathbf{m} = 0$, we arrive at the state of a spin glass, which has no analog in the ordered case. It differs from a disordered ferromagnet in that it contains blocks of spins of similar orientation, but an average orientation over the blocks does not exist. This state differs from a paramagnet in that in it the spins are disordered but "frozen", i.e., they do not move with the time and therefore give a zero value of the moment only after a double averaging, whereas in a paramagnet the moment vanishes already as a result of averaging over the Gibbs distribution, which coincides by virtue of the ergodic hypothesis with averaging with respect to the time.

The listed states of disordered spin systems can be classified by means of the parameters $\mathbf{m}$ and $q = \langle \mathbf{m}_x^2 \rangle - \langle \mathbf{m}_x \rangle^2$ of the mean and the dispersion of the field $\mathbf{m}_x$. Namely,*

$$\begin{aligned} &\mathbf{m} = 0,\ q = 0: \text{ paramagnet;} \\ &\mathbf{m} \neq 0,\ q = 0: \text{ ordered ferromagnet;} \\ &\mathbf{m} = 0,\ q \neq 0: \text{ spin glass;} \\ &\mathbf{m} \neq 0,\ q \neq 0: \text{ disordered (amorphous) ferromagnet.} \end{aligned} \tag{3.2}$$

With regard to a phase transition, in such systems it consists either of the appearance of a nontrivial random field or in a change in its nature (such a case will be described at the end of Sec.3.3). Here, we must point out that, as in ordered systems, to obtain a nonzero spontaneous magnetization it is necessary to break the symmetry of the problem by appropriate small modifications of the Hamiltonian in order to lift the degeneracy of the state of thermodynamic equilibrium; moreover, as we see in the case of spin glass, this breaking must be even more radical. As a result, we obtain an entire family of fields labeled by an index whose set of values is formed by applying to some vector all elements of the broken symmetry, i.e., the family is labeled by the index that labels the quasiaverages.

* In the general case, this picture is evidently somewhat simplified. It does not apply to the field $m_x = \Sigma g(x - x_j)$, where the random points x_j are distributed uniformly over the lattice, and $g(x)$ is an integrable function. Nevertheless, such a field can be regarded as corresponding to ferromagnetic inclusions in a paramagnetic matrix.

2. We now come to consider our models (1.5). In both the classical (s is a D-dimensional unit vector) and quantum case (**s** is the operator of a spin of magnitude s) the Hamiltonian corresponding to an interaction of such type can be written in the form

$$H=-\frac{N}{2}\sum_{k=1}^{n_1} f_k\boldsymbol{\sigma}_k^2+\frac{N}{2}\sum_{k=1}^{n_2} a_k\boldsymbol{\sigma}_{k+n_1}^2-h\sum_{x\in V} c_x\mathbf{s}_x+\tilde{H}, \tag{3.3}$$

where $\boldsymbol{\sigma}_k=N^{-1}\sum_x \alpha_x^{(k)}\mathbf{s}_x$, and

$$\tilde{H}=\frac{1}{2}\sum_{k=1}^{n_1} f_kN^{-1}\sum_{x\in V}(\alpha_x^{(k)})^2\mathbf{s}_x^2-\frac{1}{2}\sum_{k=1}^{n_2} a_kN^{-1}\sum_{x\in V}(\alpha_x^{(k+n_1)})^2\mathbf{s}_x^2.$$

Since s_x^2 is equal to 1 in the classical case and s(s + 1) in the quantum case, and we assume that $\alpha_x^{(k)}$ has moments of sufficiently high order (below, we require them to the fourth order inclusively), it follows by virtue of the ergodic theorem in the limit $N\to\infty$ that $N^{-1}\sum_{x\in V}(\alpha_x^{(k)})^2$ tend with probability 1 to finite limits $\langle(\alpha_x^{(k)})^2\rangle$ and therefore are bounded for all sufficiently large N with the same probability. This means that $\tilde{H}$ is bounded as $N\to\infty$ and therefore can be omitted, since it does not contribute to the limiting free energy. This quantity for the Hamiltonian (3.3) (without $\tilde{H}$) can be found by means of the general method developed in [7] and applied to spin systems in [11, 12]. In accordance with the basic idea of this method, we write the Hamiltonian (3.3) (without $\tilde{H}$) in the form $H_a+H_F+H_A$, where

$$H_a=-\frac{N}{2}\sum_{k=1}^{n_1} f_k\mathbf{F}_k^2+\frac{N}{2}\sum_{k=1}^{n_2} a_k\mathbf{A}_k^2+N\sum_{k=1}^{n_2} a_k\mathbf{A}_k\boldsymbol{\sigma}_{k+n_1}-N\sum_{k=1}^{n_1} f_k\mathbf{F}_k\boldsymbol{\sigma}_k-\mathbf{h}\sum_{x\in V}\mathbf{s}_xc_x, \tag{3.4a}$$

$$H_F=-\frac{N}{2}\sum_{k=1}^{n_1} f_k(\mathbf{F}_k-\boldsymbol{\sigma}_k)^2,\quad H_A=\frac{N}{2}\sum_{k=1}^{n_2} a_k(\mathbf{A}_k-\boldsymbol{\sigma}_{k+n_1})^2, \tag{3.4b}$$

and $\mathbf{A}_k$ and $\mathbf{F}_k$ are c-number parameters that will be determined below.

By virtue of Bogolyubov's inequality [7]

$$f_{H_a}+N^{-1}\langle H_A+H_F\rangle_H\leqslant f_H=-(\beta N)^{-1}\ln \operatorname{Sp} e^{-\beta H_N}\leqslant f_{H_a}+N^{-1}\langle H_A+H_F\rangle_{H_a}$$

or with allowance for the fact that in accordance with (3.4b)

$$H_F\leqslant 0,\quad H_A\geqslant 0,\quad \max_{\{A_k\}} f_{H_a}+N^{-1}\langle H_F\rangle_H\leqslant f_H\leqslant f_{H_a}+N^{-1}\langle H_A\rangle_{H_a}. \tag{3.5}$$

In this inequality, f_{H_a} and $N^{-1}\langle H_A\rangle_{H_a}$ can be readily calculated:

$$f_{H_a}=-\frac{1}{2}\sum_1^{n_1} f_k\mathbf{F}_k^2+\frac{1}{2}\sum_1^{n_2} a_k\mathbf{A}_k^2-(\beta N)^{-1}\sum_{x\in V}\varphi\left(\beta\left|\sum_1^{n_2} a_k\mathbf{A}_k\alpha_x^{(k+n_1)}-\sum_1^{n_1} f_k\mathbf{F}_k\alpha_x^{(k)}-\mathbf{h}c_x\right|\right)$$

$$N^{-1}\langle H_A\rangle_{H_a}=\frac{1}{2}\sum_1^{n_2} a_k\left[N^{-1}\sum_{x\in V}(\mathbf{A}_k-\alpha_x^{(k+n_1)}\boldsymbol{\tau}_x)^2\right]+N^{-2}\sum_1^{n_1} a_k\sum_{x\in V}(\alpha_x^{(k)})^2(\mathbf{S}^2-\boldsymbol{\tau}_x^2), \tag{3.6}$$

where

$$\varphi(r)=\ln Z(r); \tag{3.7}$$

$$Z(|\mathbf{r}|)=\begin{cases}\int ds e^{-\mathbf{rs}} \text{ in the classical case,}\\ \operatorname{Sp} e^{-\mathbf{rs}}=\dfrac{\operatorname{sh}|\mathbf{r}|(s+1/2)}{\operatorname{sh}|\mathbf{r}|/2} \text{ in the quantum case;}\end{cases}$$

$$\boldsymbol{\tau}_x=\nabla_\mu\varphi|_{\mathbf{r}=\beta\mu_x}=\langle\mathbf{s}_x\rangle_{H_a};\quad \boldsymbol{\mu}_x=\sum_{k=1}^{n_1} f_k\mathbf{F}_k\alpha_x^{(k)}-\sum_{k=1}^{n_2} a_k\mathbf{A}_k\alpha_x^{(k+n_1)}+\mathbf{h}c_x. \tag{3.8}$$

We set $\mathbf{A}_k=\langle\alpha_x^{(k+n_1)}\boldsymbol{\tau}_x\rangle$. Then in the limit $N\to\infty$, (3.6) will tend to zero by virtue of the ergodic theorem and the fact that $\langle(\alpha_x^{(k)})^2\rangle$ is finite. Then, to achieve the vanishing as $N\to\infty$ of the term $N^{-1}\langle H_F\rangle_H$ on the left-hand side in (3.5), we minimize with respect to $\{\mathbf{F}_k\}$ on the right-hand side of (3.5) in the same way as in [7], and on the left-hand side we replace the first term also by the minimum with respect to $\{\mathbf{F}_k\}$, and in the second term we set

$$\mathbf{F}_k = N^{-1}\left\langle \sum_{x\in V} \alpha_x^{(k)} \mathbf{s}_x \right\rangle_H. \tag{3.9}$$

One can see that the arguments which prove the asymptotic smallness of $N^{-1}\langle H_F\rangle_H$ under the condition (3.9) in the case $\alpha_x^{(k)}=\text{const}$ as given in [7, 11, 12] also apply in our case. The corresponding estimates have a similar form if in them the coupling constants f_k are replaced by $f_k N^{-1}\sum_{x\in V}|\alpha_x^{(k)}|$, quantities that tend with probability 1 in the limit $N\to\infty$ to $f_k\langle|\alpha_0^{(k)}|\rangle$ and are therefore finite. Therefore, as in [7], we find that the limit free energy is given by the following minimax principle:

$$f = \min_{\{F_k\}}\max_{\{A_k\}}\left\{\frac{1}{2}\sum_1^{n_1} f_k \mathbf{F}_k^2 - \frac{1}{2}\sum_1^{n_2} a_k \mathbf{A}_k^2 - \beta^{-1}\langle\varphi(\beta|\boldsymbol{\mu}_x|)\rangle\right\}. \tag{3.10}$$

Thus, the difference between the classical and quantum cases is determined solely by the form of the function $\varphi(r)$ in (3.7).* We require some properties of this function that are common to all cases:

$$\varphi(r)\geqslant 0; \tag{3.11a}$$

$$\varphi'(r)>0,\ r>0,\ \varphi'(0)=0; \tag{3.11b}$$

$$\varphi'(r)\leqslant\sigma, \tag{3.11c}$$

where $\sigma = 1$ in the classical case and $\sigma = s$ in the quantum case;

$$\varphi''(r)\geqslant 0,\ \varphi'''(r)\leqslant 0. \tag{3.11d}$$

3. We consider some special cases of the general formula (3.10). Suppose first $n_1 = 1$, $n_2 = 0$ (the case $n_1 = 0$, $n_2 = 1$ does not lead to an interesting model). Formula (3.10) here takes the form

$$f = \min_F \Phi(\mathbf{h},\mathbf{F}),\quad \Phi(\mathbf{h},\mathbf{F}) = \frac{J\mathbf{F}^2}{2} - \beta^{-1}\langle\varphi(\beta|J\alpha_x\mathbf{F}+c_x\mathbf{h}|)\rangle, \tag{3.12}$$

and the equation for the order parameter $\mathbf{F}$ and the expression for the magnetization and the susceptibility in terms of $\mathbf{F}$ have the form

$$\mathbf{F} = \left\langle \frac{\varphi'(\beta\mu)}{\mu}\alpha_x\boldsymbol{\mu}_x\right\rangle,\quad \boldsymbol{\mu}_x = J\alpha_x\mathbf{F}+c_x\mathbf{h},\quad \mu=|\boldsymbol{\mu}_x|, \tag{3.13}$$

$$\mathbf{M}(\mathbf{h}) = \left\langle \frac{\varphi'(\beta\mu)}{\mu}\boldsymbol{\mu}_x c_x\right\rangle, \tag{3.14}$$

$$\chi_{ij} = \frac{\partial M_i}{\partial h_j} = \chi_\parallel n_i n_j + \chi_\perp(\delta_{ij}-n_i n_j), \tag{3.15}$$

$$\chi_\parallel = -\frac{\partial^2 f}{\partial h^2},\quad \chi_\perp = -h^{-1}\frac{\partial f}{\partial h} \tag{3.16}$$

and $\mathbf{h} = \mathbf{n}h$, $|\mathbf{n}| = 1$. If $\mathbf{h} = 0$, then $\Phi(\mathbf{h},\mathbf{F})$ depends only on $F \equiv |\mathbf{F}|$, whose minimizing value can be found from the following equation, which is analogous to the well-known equation for the order parameter of the molecular field theory:

$$F = \langle\varphi'(\beta J F|\alpha_x|)\,|\alpha_x|\rangle. \tag{3.17}$$

A nontrivial solution $F \neq 0$ of this equation, and hence a certain phase transition appear at the temperature

$$T_c = J\varphi''(0)\langle\alpha_x^2\rangle, \tag{3.18a}$$

which with allowance for (1.6) is a linear function of the impurity concentration:

$$T_c = J\varphi''(0)\langle\gamma_x^2\rangle c. \tag{3.18b}$$

With regard to the nature of the resulting phase transition, it depends essentially on the properties of the

* In addition, this difference appears in the order of magnitude of the error $f - f_N$ as $N\to\infty$. As in ordered systems [7, 11, 12], in the classical case $f - f_N \sim N^{-\frac{1}{2}}$, and in the quantum case $f - f_N \sim N^{-2/5}$.

probability distribution of α_x. To simplify the following analysis, we shall assume that this distribution is symmetric about $\langle\alpha_x\rangle\equiv\alpha_1$, i.e., in accordance with (1.6) it has the form $\tilde{q}(\alpha-\alpha_1)$, where $\tilde{q}(\alpha)$ is an even function. Suppose first $\alpha_1 \neq 0$. By virtue of the spherical symmetry of the function φ, $\Phi(\mathbf{h}, \mathbf{F})$ for $\mathbf{h} \neq 0$ depends on $\mathbf{F}$ through the variables F_1 and F_2, which are such that $\mathbf{F}=\mathbf{n}F_1+\boldsymbol{\rho}F_2$, $|\boldsymbol{\rho}|=1$, $\boldsymbol{\rho}\perp\mathbf{n}$ and

$$F_1=\langle\varphi'(\beta\mu)\mu^{-1}(\alpha_x JF_1+c_x h)\alpha_x\rangle, \tag{3.19}$$

$$F_2=\langle\varphi'(\beta\mu)\mu^{-1}\alpha_x^2\rangle JF_2. \tag{3.20}$$

We show that the solution of this system of equations has the form $(F_1, 0)$ for all sufficiently small h, which is all that interests us.* Indeed, $\varphi(r)$ by virtue of (3.11) increases as $r\to\infty$ not faster than linearly, and therefore the values of F_1 and F_2 that minimize $\Phi(\mathbf{h}, \mathbf{F})$ as functions of h are uniformly bounded for $h \leq h_0 < \infty$. Suppose that $F_2 \neq 0$. Then (3.20) is equivalent to the equation $1=\langle\varphi'(\beta\mu)\mu^{-1}\alpha_x^2\rangle J$, by virtue of which (3.19) takes the form $\langle\varphi'(\beta\mu)\mu^{-1}\alpha_x c_x\rangle=0$. We show that this relation is impossible, i.e., that for all sufficiently small h

$$\langle\varphi'(\beta\mu)\mu^{-1}\alpha_x c_x\rangle\neq 0. \tag{3.21}$$

Indeed, denoting $JF|\alpha_x|$ by μ_0, we obtain

$$|\mu-\mu_0|^2\leqslant|\mu^2-\mu_0^2|\leqslant 2|\alpha_x|Fh+h^2. \tag{3.22}$$

It follows from (3.11) that $(\varphi'/r)'$ is bounded for $r \geq 0$, and therefore for small h the left-hand side of (3.21) differs from $\langle\varphi'(\beta\mu_0)\mu_0^{-1}\alpha_x c_x\rangle$ by an amount of order h. But

$$\langle\varphi'(\beta\mu_0)\mu_0^{-1}\alpha_x c_x\rangle = c\int_0^\infty\{\psi(\beta JF(\alpha+\alpha_1))-\psi(\beta JF(\alpha-\alpha_1))\}\tilde{q}(\alpha)\,d\alpha,$$

where $\psi(x)=\varphi'(|x|)\operatorname{sign}x$. Since $\psi(x)$ is in accordance with (3.11) a strictly increasing function, the last integral cannot be zero. The resulting contradiction proves that $F_2 = 0$.

Thus, as in the ordered case, if the Hamiltonian for $h = 0$ is spherically symmetric then the order parameter is parallel to the field when the field is switched on. In accordance with (3.14), the magnetization has a similar property, and by virtue of the fact we have just proved it is also nonzero for $h = 0$:

$$\mathbf{M}(0)=m\mathbf{n},\quad m=\langle\psi(\beta JF_1\alpha_x)c_x\rangle, \tag{3.23}$$

where F_1 is the solution, taken at $h = 0$, of the equation

$$F_1=\langle\psi(\beta[J\alpha_x F_1+c_x h])\alpha_x\rangle, \tag{3.24}$$

which minimizes the function $\Phi(\mathbf{h}, \mathbf{F})$ in (3.12). It is unique, positive, and for $h = 0$ coincides with the solution (3.17). In addition, it follows from (3.16) that $\chi_\perp = m/h$ as $h\to 0$, and

$$\chi_\parallel|_{h=0}=\beta\left\langle c_x\varphi''(\beta JF_1\alpha_x)\left(J\alpha_x\frac{\partial F_1}{\partial h}+c_x\right)\right\rangle. \tag{3.25}$$

It is readily seen that $F_1|_{h=0}=F$ in the neighborhood of T_c is such that

$$F^2=\frac{6[\varphi''(0)\alpha_2]^2}{|\varphi^{\mathrm{IV}}(0)|\alpha_4 J}\max\{(T_c-T),0\}, \tag{3.26}$$

where $\alpha_k=\langle\alpha^k\rangle$. Hence and from (3.24) we can find that in the case $\langle\alpha\rangle=\eta_1\neq 0$ as $T\to T_c$ the term containing $\partial F_1/\partial h$ is the main term in (3.25) and it leads to the usual Curie–Weiss law for χ,

$$\chi_\parallel\simeq\alpha_1^2/\alpha_2|T-T_c|A_\pm\quad T\to T_c\pm 0,$$

where in the case of the Ising model and the quantum Heisenberg model with spin $^1/_2$ $A_+=2A_-=1$.

Thus, in the case of asymmetrically distributed $\alpha_x(\langle\alpha_x\rangle\neq 0)$ we obtain a theory analogous to the molecular field theory for ferromagnets, and it is therefore natural to regard such a model as a model of a disordered (amorphous) ferromagnet. It is also this in the sense of the classification (3.2), which follows from the form of the random field of the spontaneous magnetization:

$$c_x\langle\mathbf{s}_x\rangle_G|_{h=0}\equiv\mathbf{m}_x=c_x\psi(\beta JF\alpha_x)\mathbf{n}, \tag{3.27}$$

* The given arguments are valid under the assumption $T < T_c$. But if $T > T_c$, then it is readily seen from (3.11) and (3.20) that the assumption $F_2 \neq 0$ contradicts (3.18).

where $\mathbf{n}$ is an arbitrary D-dimensional unit vector ($n = \pm 1$ for $D = 1$). In order to obtain (3.27), it is necessary to generalize the method of proof developed in [7] (see also [11]) of asymptotic proximity of the thermodynamic mean values constructed on the basis of the original model Hamiltonian and the approximating Hamiltonian (3.4a). Such a generalization can be made and leads to a result that, as in the ordered case, asserts that the corresponding mean values can be calculated from the approximating Hamiltonian in which the parameters $\mathbf{F}_k$ and $\mathbf{A}_k$ are chosen as the solutions of the equations of the generalized minimax principle (3.10) (for $n_1 = 1$, $n_2 = 0$ this is Eq. (3.24)).

Taking into account the properties of the random variables α_x and c_x and the ergodic theorem, we can readily show that the field (3.27) satisfies the relation (3.1), and by virtue of (3.21) $\langle \mathbf{m}_x \rangle = m\mathbf{n} \neq 0$. In addition, since α_x and c_x have by hypothesis a distribution different from a delta function, $q = \langle |\mathbf{m}_x|^2 \rangle - \langle \mathbf{m}_x \rangle^2 \neq 0$.

Now suppose that $\alpha_1 = 0$ and therefore positive and negative values of the exchange integral are now equally probable. It can be seen from (3.14) and (3.15) that in this case $\mathbf{M}|_{h=0} = 0$ for $T < T_c$ as well, and the magnetic susceptibility becomes a symmetric tensor (a distinguished direction, which is defined by the spontaneous magnetization in a ferromagnet, is here absent):

$$\chi_{ik}|_{h=0} = \chi\delta_{ik}, \quad \chi = \beta\langle c_x \varphi''(\beta J F \alpha_x) \rangle. \tag{3.28}$$

Hence and from (3.26) it follows that χ is a continuous function of the temperature in the neighborhood of T_c, but its derivative with respect to T has a discontinuity at this point:

$$-\frac{\partial \chi}{\partial T}\Big|_{T=T_c+0} = \beta_c^2 \varphi''(0) c, \quad -\frac{\partial \chi}{\partial T}\Big|_{T=T_c-0} = -\beta_c^2 \varphi''(0) c\left[1 - \frac{3\alpha_2^2}{\alpha_4}\right]. \tag{3.29}$$

In accordance with what we have said in the introduction and Sec. 3.1, this state of the spin system is a spin glass. This conclusion agrees with the classification (3.2) since, as one can show, the random field of the spontaneous magnetization here also has the form (3.27), but now, because $\psi(x)$ is odd and the distribution symmetric, $\alpha_x \langle \mathbf{m}_x \rangle = 0$. With regard to the dispersion of the field, it is here, as in a disordered ferromagnet, nonzero and equal to

$$\langle \varphi'^2(\beta J F \alpha_x) c_x^2 \rangle.$$

In the case of the Ising model and the quantum Heisenberg model with spin $\frac{1}{2}$ the following relation between χ and q also holds:

$$\chi = \chi_0[1 - (\sigma^2 c)^{-1} q],$$

where $\chi_0 = c\beta\sigma$ is the susceptibility of the disordered system of noninteracting spins, and σ is the quantity in (3.11c). An analogous relation was obtained in [3].

It should be noted that to obtain a nonzero field $\mathbf{m}_x$ in spin glass it is not sufficient to include an infinitesimally small constant field; for in the case of symmetrically distributed α_x the right-hand side of Eq. (3.24) is an odd function of F_1 even for $h \neq 0$, and therefore every solution F_1 of (3.24) is accompanied by the solution $-F_1$. This means that the thermodynamic degeneracy is not completely lifted by the constant field, and therefore in the calculation of the Gibbs mean of $\mathbf{s}_x$, when it is necessary to sum over all degenerate states, one obtains a zero result since states corresponding to the solutions $\pm F_1$ enter with equal weights.* Thus, because of the higher symmetry of the problem (the additional symmetry is due to the invariance under the substitution $\alpha_x \to -\alpha_x$) to lift the degeneracy it is necessary to use perturbations that differ from an expression of the form $\sum \mathbf{h}\mathbf{s}_x$. Thus, for example, one can take the addition $\sum \mathbf{h}_x \mathbf{s}_x$, where $\mathbf{h}_x$ is a random field that is necessarily statistically related to α_x (otherwise the right-hand side of (3.24) remains odd with respect to F_1). For example, $\mathbf{h}_x = \varepsilon\alpha_x$. Another possible way of lifting the degeneracy is to return to asymmetrically distributed α_x, for example, to variables of the form $\alpha_x + \varepsilon$, where α_x is as before symmetric. One can show that in both cases the function analogous to $\Phi(\mathbf{h}, \mathbf{F})$ in (3.12) has a minimum with respect to F_1 at a unique point, and this leads to the nonzero field (3.27).

Note that this formula shows that the radius r_m of the statistical correlation of the field $\mathbf{m}_x$, which determines the size of the blocks of spins of similar orientation, coincides in order of magnitude in the considered model with the correlation radius r_α of the random variables α_x, i.e., with the distance at which $\langle \alpha_x \alpha_y \rangle - \langle \alpha_x \rangle \langle \alpha_y \rangle$ decreases appreciably as $|x - y| \to \infty$. With regard to r_α, it is a parameter of the theory,

* One can show that in the case of a function $q(\alpha)$ of compact support this circumstance leads to analyticity of $M(h)$ in the neighborhood of the origin.

and the fact that $r_\alpha \sim r_m$ could perhaps be used for its experimental determination.

One further interesting relation follows from (3.27). In the exactly solvable case of the Ising model, there is [5] a connection between the asymptotic behavior of the Gibbs correlation function $\langle s_x s_y \rangle_G$ and the spontaneous magnetization **M**:

$$\lim_{|x-y|\to\infty} \langle s_x s_y \rangle_G = \mathbf{M}^2. \tag{3.30}$$

It can be shown that if the limit is understood here in a somewhat more general sense then a relation of such type will also hold in our model. Namely, with probability 1

$$\lim_{V\to\infty} N^{-2} \sum_{x,y\in V} c_x c_y \langle s_x{}^i s_y{}^j \rangle = n_i n_j m^2. \tag{3.31}$$

Since the left-hand sides of (3.30) or (3.31) are usually regarded as a measure of the thermodynamic long-range order in the system, (3.30) shows that in a disordered ferromagnet long-range order exists in this sense, as in an ordered ferromagnet, while it is absent in spin glass.

We give also formulas for the specific heat of the system. From (3.12) we find that irrespective of the symmetry properties of the random variables α_x, at $h = 0$

$$C_V = J\beta^2 F \frac{\partial F}{\partial\beta} = \frac{\beta^2 J^2 \langle \varphi''(\beta J F \alpha_x)\alpha_x{}^2\rangle}{1 - J\beta\langle \varphi''(\beta J F\alpha_x)\alpha_x{}^2\rangle}, \tag{3.32}$$

where F is the solution (3.17). Using the first of these equations and (3.26), we find that

$$C_V(T_c+0) = 0,\ C_V(T_c-0) = 3\alpha_2{}^2\varphi''(0)/\alpha_4\varphi^{IV}(0),$$

i.e., in the neighborhood of T_c the specific heat behaves in the same way as in the ordinary molecular field theory [5]. When allowance is made for the form (1.6) of the probability density α_x as a function of the concentration, the second of equations (3.32) leads to the following relations as $T \to 0$:

$$\lim_{T\to 0} C_V(T) = c(D-1)/2$$

in the classical model, and

$$\lim_{T\to 0} T^{-1} C_V(T) = \frac{2}{3(2s+1)} \frac{\pi^2 q(0)}{J} \left(\int |\alpha| q(\alpha)\, d\alpha\right)^{-1} \tag{3.33}$$

in the quantum model. In the case of the Ising model, which here resembles in many respects the quantum model, (3.33) also holds, but with coefficient $1/12$ instead of $2/3(2s+1)$.

We now describe briefly the situation that arises in the next most complicated model $n = 2$ in (3.12).* Here, the following possibilities arise.

A. $n_1 = 2$, $n_2 = 0$. In such a model, there are two phase transitions at the temperatures $T_{c_1} = f_1\varphi''(0)\langle(\alpha_x^{(1)})^2\rangle$ and $T_{c_2} = \langle(\alpha_x^{(2)})^2\varphi''(\beta_{2c})\alpha_x^{(1)} F_1(\beta_{2c})\rangle < T_{c_1}$, where $F_1(\beta)$ is a solution of Eq. (3.24) (we assume that $\langle(\alpha_x^{(1)})^2\rangle > \langle(\alpha_x^{(2)})^2\rangle$). The first of the transitions is one from a paramagnetic state to a state with nonzero random field m_x, and this will be either a disordered ferromagnet (if $\alpha_x^{(1)}$ is asymmetric) or a spin glass (if $\alpha_x^{(1)}$ is symmetric). In the second transition, the nature of the random field m_x changes. If $\alpha_x^{(1)}$ is symmetric, this is a transition from a spin glass to a disordered ferromagnet in the case of asymmetric $\alpha_x^{(2)}$, so that as a result of the transition $\langle m_x\rangle \equiv m$ becomes nonzero. If $\alpha_x^{(2)}$ is symmetric, then this is a transition from one phase of spin glass type to another of the same type, but with a different value of q. But if $\alpha_x^{(1)}$ is asymmetric, then for $T = T_{c_2}$ there is always a phase transition from one disordered ferromagnetic phase to another. The thermodynamic quantities behave qualitatively in the second transition in the same way as in the first. Thus, this model demonstrates the possibility of several transitions between different states of the spin glass, these succeeding one another as the temperature decreases.

B. $n_1 = 1$, $n_2 = 1$. We consider only the case of symmetric $\alpha_x^{(1)}$ and $\alpha_x^{(2)}$. It can be shown that in

* As this paper was being prepared for press, we were acquainted with the note [13], in which Luttinger considers a special case of this model in which each of the statistically independent variables $\alpha_x^{(1)}$ and $\alpha_x^{(2)}$ each takes two specially chosen values. The method used in [13] differs fundamentally from ours and does not make it possible to consider more complicated distributions.

this most interesting case for $\mathbf{h} = 0$ the model is analogous to the case considered above with $n_1 = 1$, $n_2 = 0$. This comes about because, as one can show, the order parameter A_1 is here zero, i.e., the positive-definite term in the Hamiltonian does not contribute to the thermodynamics. When the external field $\mathbf{h}$ is switched on, this contribution appears ($\mathbf{A}_1$ is no longer zero), but it does not lead to any singularities.

C. $n_1 = 0$, $n_2 = 2$. In zero field, such a model is equivalent to a system of noninteracting spins. This fact agrees with the ordered case, in which the positive (antiferromagnetic) interaction does not contribute to the macroscopic properties [7].

The results we have obtained are also true with obvious modifications in the formulations in the general case of any finite number of terms in (1.5) in models with $D > 1$, both classical and quantum.

One can consider similarly anisotropic models in which each term $J_k\alpha_x^{(k)}\alpha_y^{(k)}s_xs_y$ in the Hamiltonian is replaced by $\sum_1^D J_{kl}\alpha_{x,l}^{(k)}\alpha_{y,l}^{(k)}s_x^l s_y^l$, where s_x^l, $l = 1, \ldots, D$, are the components of the spin at the point x. In the special case $n = 1$, the expressions for the thermodynamic functions can be obtained from Eqs. (3.25), (3.29), and (3.32) by understanding α_x^2 in them as $D^{-1}\sum_{l=1}^D \alpha_{x,l}^2$. The number of nontrivial fields is here equal to 2^D by virtue of the lower symmetry of the Hamiltonian.

Finally, the results obtained in the present paper can also be applied to disordered systems in which electric dipole moments play the role of spins (in this connection, see [14]).

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