Multiplicities of K-types in principal series

Alessandra Pantano

joint work with Dan Barbasch

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INTRODUCTION/MOTIVATION

Introduction

find the unitary dual of split $G_{\mathbb{R}}$	$\longrightarrow$	discuss unitarity of Langlands quotients of principal series $J_P(\delta,\nu) \rightsquigarrow P = MAN$	$\longrightarrow$	signature of some Hermitian operators $A_\mu(\delta,\nu)$ $\mu \in \widehat{K}, \delta \in \widehat{M}, \nu \in a^*_{\mathbb{C}}$

The intertwining operator $A_\mu(\delta,\nu)$ acts on $\mathbf{Hom}_M(\boldsymbol{\delta},\boldsymbol{\mu})$.

PROBLEM Understand the representation of $W(\delta)$ (= the stabilizer of δ in W) on the space $\mathbf{Hom}_M(\boldsymbol{\delta},\boldsymbol{\mu})$, $\forall \delta \in \widehat{M}$, $\mu \in \widehat{K}$.

Spherical unitary dual

spherical

unitary dual

of split $G(\mathbb{R})$

?

use spherical petite K-types

to prove that

$J(\nu)_{\mathbb{R}}$ unit. $\Rightarrow$ $J(\nu)_{\mathbb{Q}_p}$ unit.

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Barbasch–Vogan

spherical

unitary dual

of split $G(\mathbb{Q}_p)$

✓

$\Uparrow$

candidates: $J(\nu)_{\mathbb{R}}$

$\Uparrow$

candidates: $J(\nu)_{\mathbb{Q}_p}$

$J(\nu)_{\mathbb{R}}$ unitary $\Leftrightarrow$

$\boxed{A_\mu(\nu)} \geq 0, \ \forall \mu \in \widehat{K}$

$J(\nu)_{\mathbb{Q}_p}$ unitary $\Leftrightarrow$

$\boxed{A_\psi(\nu)} \geq 0, \ \forall \psi \in \widehat{W}_{relev}$

Non-spherical unitary dual

non-spher. unitary dual of split $G_{\mathbb{R}}$?	use non-spherical petite K-types to investigate whether $J^G(\delta,\nu)$ unit $\Rightarrow J^{G_0(\delta)}(\nu_0)$ unit -> *Barbasch–Pantano*	spherical unitary dual of split $G_0(\delta)$ ✓
$\Uparrow$	$\uparrow$	$\Uparrow$
candidates: $J^G(\delta,\nu)$	$\rightsquigarrow$ define G_0^δ	*candidates*: $J^{G_0^\delta}(\nu_0)$

$J^G(\delta,\nu)$ unitary $\Leftrightarrow$ $\boxed{\boldsymbol{A_\mu(\delta,\nu)}} \geq 0$ $\forall \mu \in \widehat{K}$ $\dashrightarrow$ $\boxed{\mathbf{Hom}_M(\delta,\nu)}$ $\dashleftarrow$ $J^{G_0^\delta}(\nu_0)$ unitary $\Leftrightarrow$ $\boxed{\boldsymbol{A_\psi(\nu)}} \geq 0$ $\forall \psi \in \widehat{W_0}$ relevant

Two projects

BIG PROJECT

Find an inductive algorithm

to compute

the $W(\delta)$-representation

$\mathbf{Hom}_M(\boldsymbol{\delta}, \boldsymbol{\mu})$

$\longrightarrow$ *July*

SMALL PROJECT

Find an inductive algorithm

to compute

$\mathbf{dim}[\mathbf{Hom}_M(\boldsymbol{\delta}, \boldsymbol{\mu})]$

$\longrightarrow$ *today*

Plan of the talk

- Standard Notation
- Multiplicities of K-types in principal series
- Some easy examples (*linear case*)
- Non-linear case (*what we know...*)
- An inductive algorithm to compute multiplicities
- Generalization

PART 1

- **Standard Notation**
- Multiplicities of K-types in principal series
- Some easy examples (*linear case*)
- Non-linear case (*what we know...*)
- An inductive algorithm to compute multiplicities
- Generalization

Notation

- $\boxed{G}$ a real reductive Lie group ← *split group*
- $\boxed{K}$ the maximal *compact* subgroup of G
- $\boxed{K\text{-types}}$ the irreducible representations of K

$$\mu = \sum a_j \omega_j, \text{ with } a_j \geq 0 \text{ and } \omega \text{ fundamental}$$

- $\boxed{\theta}$ a Cartan involution on $\mathfrak{g}$
- $\boxed{\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}}$ the Cartan decomposition of $\mathfrak{g}$
- $\boxed{\mathfrak{a}}$ a maximal abelian subspace of $\mathfrak{p}$, $\boxed{A = \exp(\mathfrak{a})}$
- $\boxed{M = Z_K(\mathfrak{a})}$ ← *finite subgroup of K*
- $\boxed{P = MAN}$ a minimal parabolic subgroup of G

Minimal Principal Series

parameters $\begin{cases} P = MAN & \text{minimal parabolic subgroup of } G \\ (\delta, V^\delta) & \text{irreducible representation of } M \\ \nu\colon \mathfrak{a} \longrightarrow \mathbb{C} & \text{dominant character of } A \end{cases}$

principal series $\boxed{\boldsymbol{I_P(\delta,\nu)}} = \mathrm{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes triv)$

G acts by left translation on the space of functions

$\{F\colon G \to V^\delta \colon F\,|_K \in L^2,\ F(xman) = e^{-(\nu+\rho)log(a)}\delta(m)^{-1}F(x),\ \forall\, man \in P\}$

PART 2

- Standard Notation
- **Multiplicities of K-types in principal series**
- Some easy examples (*linear case*)
- Non-linear case (*what we know...*)
- An inductive algorithm to compute multiplicities
- Generalization

Multiplicities of K-types in Principal Series

Which irreducible representations μ of K appear in the principal series $I_P(\delta, \nu)$, and with what multiplicities?

A reformulation of this problem

The multiplicity of a K-type μ in $I_P(\delta, \nu)$ is defined by

$$\boldsymbol{m(\mu,\, I_P(\delta,\nu)) = \dim\left[\mathrm{Hom}_K(\mu,\, \mathrm{Res}_K I_P(\delta,\nu))\right]}$$

By Frobenius reciprocity, it is independent of the parameter ν:

$$\boldsymbol{m(\mu,\, I_P(\delta,\nu)) = m(\delta,\, \mu) = \dim\left[\mathrm{Hom}_M(\delta,\, \mathrm{Res}_M \mu)\right]}$$

$\Rightarrow$ *We need to study the restriction of K-types to M*

PART 3

- Standard Notation
- Multiplicities of K-types in principal series
- **Some easy examples** (*linear case*)
- Non-linear case (*what we know...*)
- An inductive algorithm to compute multiplicities
- Generalization

The example of $SL(2,\mathbb{R})$

- $G = SL(2,\mathbb{R})$, $K = SO(2,\mathbb{R})$, $M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}_2$
- $\widehat{K} = \mathbb{Z}$, $\widehat{M} = \{trivial,\ sign\}$
- $\mathrm{Res}_M(\mu_n) = \begin{cases} trivial & \text{if } n \text{ is even} \\ sign & \text{if } n \text{ is odd} \end{cases}$

$$\Rightarrow \boxed{m(\mu_{2l}, I_P(\delta,\nu)) = \begin{cases} 1 & \text{if } \delta \text{ is } trivial \\ 0 & \text{if } \delta \text{ is } sign \end{cases}}$$

and

$$\boxed{m(\mu_{2l+1}, I_P(\delta,\nu)) = \begin{cases} 0 & \text{if } \delta \text{ is } trivial \\ 1 & \text{if } \delta \text{ is } sign \end{cases}}$$

The example of $SL(3,\mathbb{R})$

- $G = SL(3,\mathbb{R})$, $K = SO(3,\mathbb{R})$
- $M = \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) \colon \epsilon_i = \pm 1,\ \Pi\epsilon_i = 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\widehat{K} = \{\mathcal{H}_n\}_{n\in\mathbb{N}} = \{p(x,y,z) \colon$ harmonic, homog. of degree n$\}$
- $\widehat{M} = \{triv \otimes triv,\ triv \otimes sign,\ sign \otimes triv,\ sign \otimes sign\}$
- $\mathcal{H}_{2l}\,|_M = [tr \otimes tr]^{l+1} \oplus [tr \otimes sign]^l \oplus [sign \otimes tr]^l \oplus [sign \otimes sign]^l$

$$\Rightarrow \boxed{m(\mathcal{H}_{2l}, I_P(\delta,\nu)) = \begin{cases} l+1 & \text{if } \delta = tr \otimes tr \\ l & \text{otherwise} \end{cases}}$$

There are similar formulas for $\mathcal{H}_{2l+1}$

Non-linear groups

Suppose that

- $\mathbb{G}$: a simple, connected and simply connected real reductive algebraic group
- G: the split real form of $\mathbb{G}$
- $\widetilde{G}$: the (unique) two-fold cover of G

then

$\widetilde{G}$ is non-linear and $\widetilde{M}$ is non-abelian

Restricting representations from $\widetilde{K}$ to $\widetilde{M}$ is obviously harder!

PART 4

- Standard Notation
- Multiplicities of K-types in principal series
- Some easy examples (*linear case*)
- **Non-linear case** (*what we know about* $\widetilde{M}\ldots$)
- An inductive algorithm to compute multiplicities
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Notation

For each root α, we can choose a Lie algebra homomorphism

$$\phi_\alpha \colon \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}$$

such that

$$\boxed{Z_\alpha} = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k} = \mathrm{Lie}(K).$$

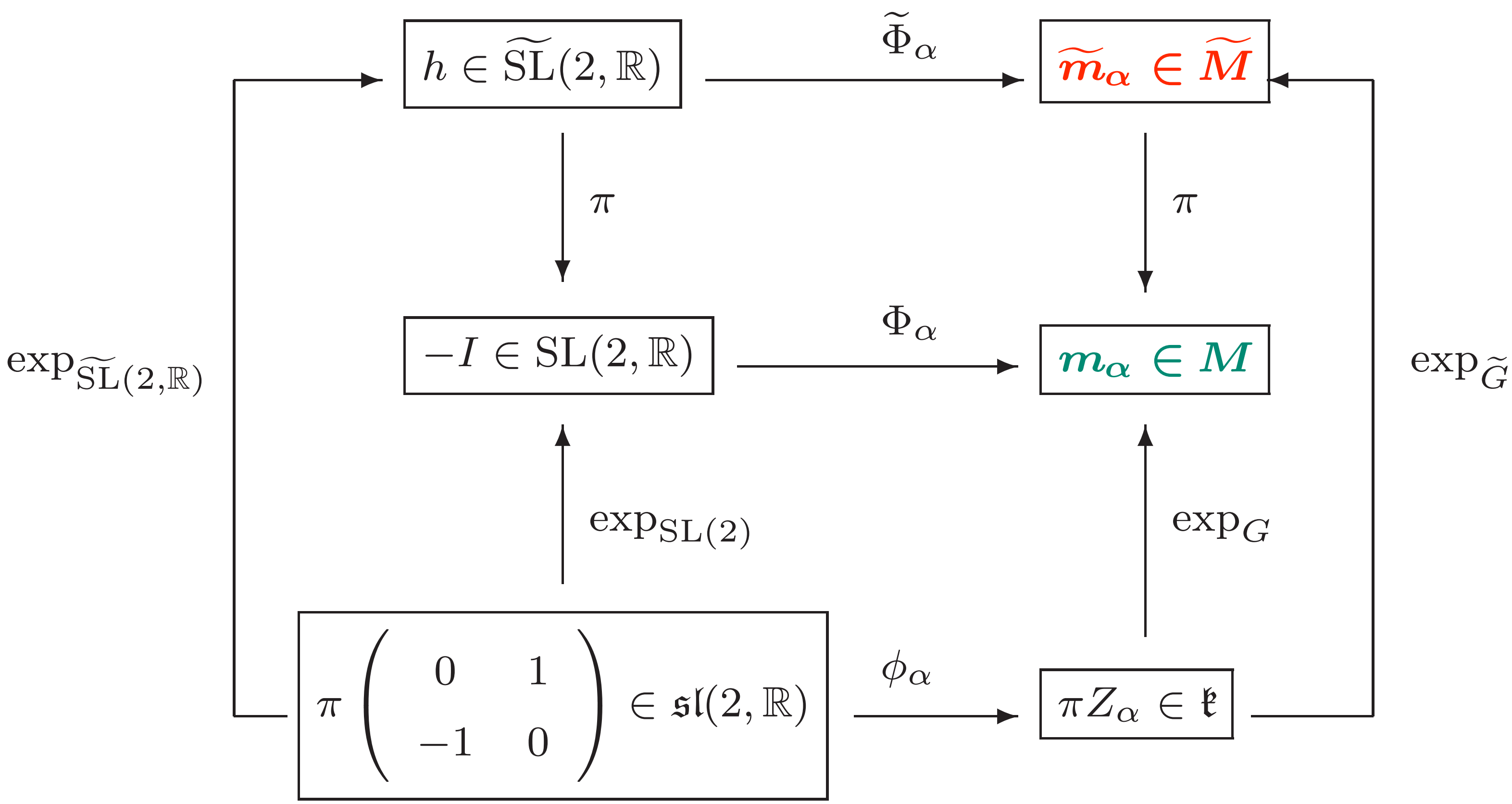
Exponentiating ϕ_α, we obtain

$$\Phi_\alpha \colon SL(2,\mathbb{R}) \to G \qquad \widetilde{\Phi}_\alpha \colon \widetilde{SL}(2,\mathbb{R}) \to \widetilde{G}.$$

*Definition: α is **metlapectic** if $\widetilde{\Phi}_\alpha$ does not factor to $SL(2,\mathbb{R})$.*

If G is not of type G_2, then **metaplectic $\Leftrightarrow$ long**,
if G is of type G_2, then all roots are metaplectic.

More notation: $\widetilde{m}_\alpha = \exp_{\widetilde{G}}(\pi Z_\alpha)$ and $m_\alpha = \exp_G(\pi Z_\alpha)$



Structure of $\widetilde{M}$

- **GENERATORS**: $\{\widetilde{m}_\alpha\}_{\alpha \text{ simple}}$

- **RELATIONS**: $\widetilde{m}_\alpha^2 = \begin{cases} -I & \text{if } \alpha \text{ is metaplectic} \\ +I & \text{otherwise} \end{cases}$

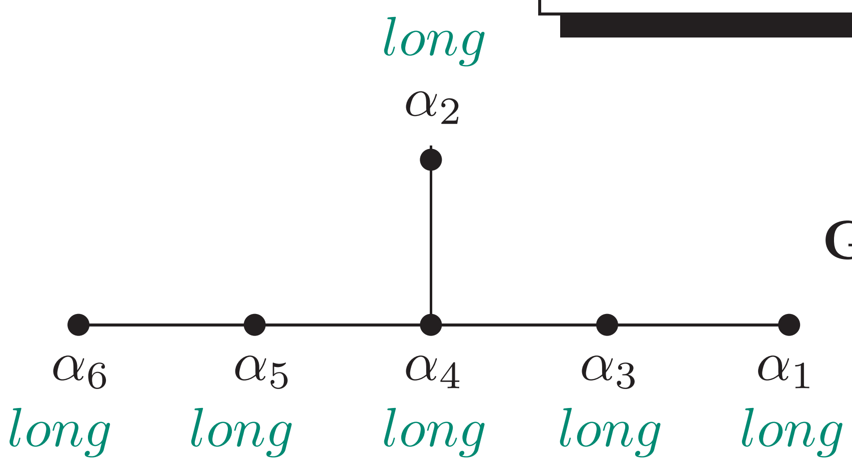
  and $\{\widetilde{m}_\alpha, \widetilde{m}_\beta\} = \begin{cases} (-I)^{\langle \alpha, \check{\beta} \rangle} & \text{if } \alpha \text{ and } \beta \text{ are } \textit{both} \text{ metaplectic} \\ +I & \text{otherwise.} \end{cases}$

- **ELEMENTS**: Choose an ordering of the simple roots. Every element of $\widetilde{M}$ can be written uniquely in the form

$$\varepsilon\, \widetilde{m}_{\alpha_1}^{n_1}\, \widetilde{m}_{\alpha_2}^{n_2} \ldots \widetilde{m}_{\alpha_r}^{n_r}$$

  with $\varepsilon = \pm 1$, and $n_j = 0$ or 1.

Example: $\widetilde{M} \subset \widetilde{E}_6$



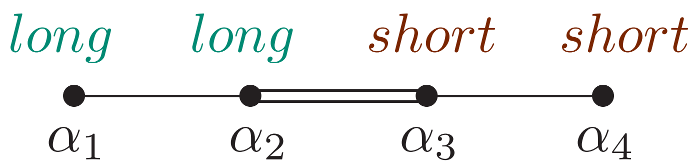
GENERATORS: $\{\widetilde{m}_{\alpha_i}\}_{i=1\ldots 6}$

RELATIONS: $\widetilde{m}_{\alpha_i}^2 = -I$ for all $i = 1\ldots 6$, and

$$\{\widetilde{m}_\alpha, \widetilde{m}_\beta\} = (-I)^{\langle \alpha_i, \check{\alpha}_j \rangle} = \begin{cases} (-I) & \text{if } \alpha_i \text{ and } \alpha_j \text{ are } \textit{adjacent} \\ (+I) & \text{otherwise.} \end{cases}$$

CENTER: $Z(\widetilde{M}) = \{\pm I\} \simeq \mathbb{Z}_2$

Example: $\widetilde{M} \subset \widetilde{F}_4$



GENERATORS: $\{\widetilde{m}_{\alpha_i}\}_{i=1\ldots4}$

RELATIONS: $\widetilde{m}_\alpha^2 = \begin{cases} -I & \text{if } \alpha \text{ is long} \\ +I & \text{if } \alpha \text{ is short} \end{cases}$

and $\{\widetilde{m}_\alpha, \widetilde{m}_\beta\} = \begin{cases} (-I) & \text{if } \alpha \text{ and } \beta \text{ are } \textit{both long} \\ (+I) & \text{otherwise.} \end{cases}$

CENTER: $Z(\widetilde{M}) = \langle -I, \widetilde{m}_{\alpha_3}, \widetilde{m}_{\alpha_4} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Representations of $\widetilde{M}$

$\widetilde{M}$ is a cover of the abelian group M. There is an exact sequence

$$1 \to \{\pm I\} \to \widetilde{M} \to M \to 1.$$

A repr. of $\widetilde{M}$ is called genuine if $(-I)$ does not act trivially

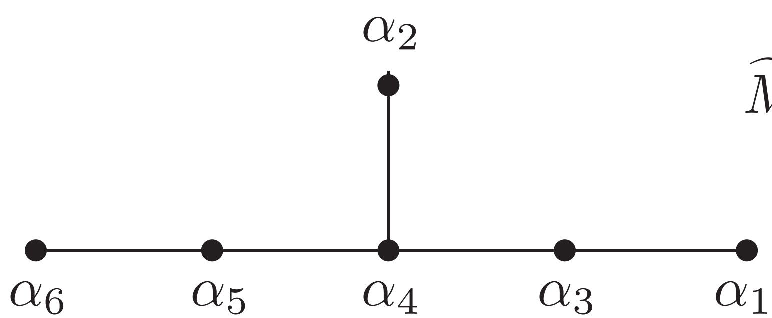
- **The non-genuine representations of $\widetilde{M}$ have dim. 1. They are determined by the value of $\delta(\widetilde{m}_{\alpha_i}) = \pm 1$**
- **The genuine repr.s of $\widetilde{M}$ have dim. $n = |\widetilde{M}/Z(\widetilde{M})|^{\frac{1}{2}}$. They are determined by the restriction to $Z(\widetilde{M})$**

$$\{\text{genuine repr.s of } \widetilde{M}\} \quad \leftrightarrow \quad \{\text{genuine characters of } Z(\widetilde{M})\}$$

$$\boxed{\delta} \quad \to \quad \boxed{\lambda} \text{ s.t. } \operatorname{Res}\delta = \lambda^{\oplus n}$$

$$\boxed{\delta} \text{ s.t. } \operatorname{Ind}\lambda = \pi^{\oplus n} \quad \leftarrow \quad \boxed{\lambda}$$

Example: representations of $\widetilde{M} \subset \widetilde{E}_6$



$$\widetilde{M} = \{\pm \widetilde{m}_{\alpha_1}^{n_1} \widetilde{m}_{\alpha_2}^{n_2} \ldots \widetilde{m}_{\alpha_6}^{n_6} : n_i = 0, 1\}$$

$$Z(\widetilde{M}) = \{\pm I\}$$

Every **non-genuine** representation is one-dimensional, and is determined by the 6-upla $[\delta(\widetilde{m}_{\alpha_1}), \ldots, \delta(\widetilde{m}_{\alpha_6})]$.
For $\delta(\widetilde{m}_{\alpha_i}) = \pm 1$, there are 2^6 distinct non-genuine representations.

The group $Z(\widetilde{M})$ has one genuine repr. χ_g, given by $\chi_g(-I) = -1$.
Hence $\widetilde{M}$ has *only one* **genuine** repr. δ_g. The dimension of δ_g is

$$|\widetilde{M}/Z(\widetilde{M})|^{\frac{1}{2}} = \sqrt{2 \cdot 2^6/2} = 8.$$

To compute the character of δ_g, we use the fact $8\delta_g = \mathrm{Ind}_{Z(\widetilde{M})}^{\widetilde{M}} \chi_g$.

PART 5

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An inductive algorithm to compute multiplicities

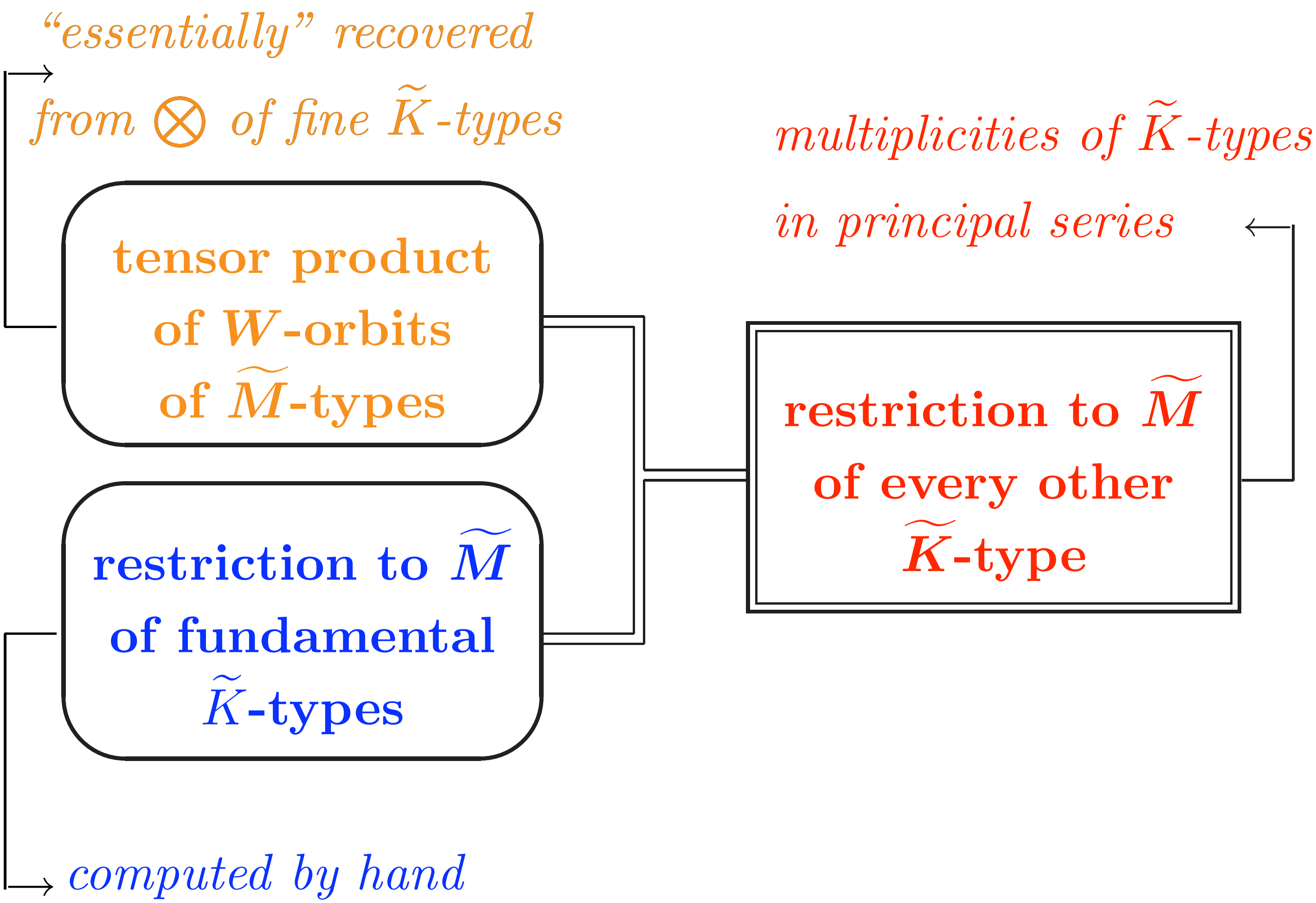
INPUT

tensor product of W-orbits of $\widetilde{M}$-types

restriction to $\widetilde{M}$ of fundamental $\widetilde{K}$-types

OUTPUT

restriction to $\widetilde{M}$ of every other $\widetilde{K}$-type



A VERY COOL FACT: in order to restrict $\widetilde{K}$-types to $\widetilde{M}$, we need very little information about the actual repr.s of $\widetilde{M}$

Computing the restriction of a $\widetilde{K}$-type μ to $\widetilde{M}$

(by induction on level and lexicographical order)

- μ embeds in a tensor product of fundamental representations
- we can write $\mu = \mu' + \omega$, with ω fundamental and μ' lower in the induction

$$\boxed{\mu' \otimes \omega = \mu + (\text{lower terms})} \qquad (\bigstar)$$

- The restriction of μ' and ω to $\widetilde{M}$ are known (*by induction*)
- The restriction of $\mu' \otimes \omega$ to $\widetilde{M}$ is computed using the table of tensor product of W-orbits of $\widetilde{M}$-types (*base of induction*)
- Equation $(\bigstar)$ gives $\mathrm{Res}_{\widetilde{M}}\, \mu$ (*by comparison*)

An example

Let $\widetilde{G} = \widetilde{F}_4$, $\widetilde{K} = SP(1) \times SP(3)$ and $\mu = (0|200)$.

$$\underbrace{(0|200)}_{\boldsymbol{\mu}} = \underbrace{(0|100)}_{\boldsymbol{\mu'}} + \underbrace{(0|100)}_{\boldsymbol{\omega}} \quad \Rightarrow \quad \mu \hookrightarrow \mu' \otimes \omega$$

lower in induction fundamental

Restriction to $\widetilde{M}$ gives:

$$\underbrace{(0|100) \otimes (0|100)}_{\bar{\delta}_6 \otimes \bar{\delta}_6} = \underbrace{(0|200)}_{\boxed{?}} \oplus \underbrace{(0|110)}_{2\delta_0 \oplus \bar{\delta}_{12}} \oplus \underbrace{(0|000)}_{\delta_0}.$$

We know that $\bar{\delta}_6 \otimes \bar{\delta}_6 = 3\delta_0 \oplus 3\bar{\delta}_3 \oplus 2\bar{\delta}_{12}$. Hence

$$\mathrm{Res}(0|200) = 3\bar{\delta}_3 \oplus \bar{\delta}_{12}$$

by comparison.

BASE OF INDUCTION

for double covers of exceptional groups

The two-fold cover of E_6

- $\widetilde{G} = \widetilde{E}_6$
- $\widetilde{K} = Sp(4)$

W-orbit of $\widetilde{M}$-types	dim.	fine $\widetilde{K}$-type	W^0_δ	$W(\delta)$
δ_1	1	(0)	$W(E_6)$	$W(E_6)$
δ_8	8	w_1	$W(E_6)$	$W(E_6)$
$\bar{\delta}_{27}$	$27 \cdot 1$	w_2	$W(D_5)$	$W(D_5)$
$\bar{\delta}_{36}$	$36 \cdot 1$	$2w_1$	$W(A_5 A_1)$	$W(A_5 A_1)$

fundam. $\widetilde{K}$-type	$\#\delta_1$	$\#\delta_8$	$\#\bar{\delta}_{27}$	$\#\bar{\delta}_{36}$
w_1	0	1	0	0
w_2	0	0	1	0
w_3	0	6	0	0
w_4	6	0	0	1

$\otimes$	δ_8	$\bar{\delta}_{27}$	$\bar{\delta}_{36}$
δ_8	$\delta_1 + \bar{\delta}_{27} + \bar{\delta}_{36}$	$27\delta_8$	$36\delta_8$
$\bar{\delta}_{27}$	$27\delta_8$	$27\delta_1 + 10\bar{\delta}_{27} + 12\bar{\delta}_{36}$	$16\bar{\delta}_{27} + 15\bar{\delta}_{36}$
$\bar{\delta}_{36}$	$36\ \delta_8$	$16\bar{\delta}_{27} + 15\bar{\delta}_{36}$	$36\delta_1 + 20\bar{\delta}_{27} + 20\bar{\delta}_{36}$

The two-fold cover of E_8

- $\widetilde{G} = \widetilde{E}_8$
- $\widetilde{K} = Spin(16)$

W-orbit of $\widetilde{M}$-types	dim.	fine $\widetilde{K}$-type	W_δ^0	$W(\delta)$
δ_0	1	(0)	$W(E_8)$	$W(E_8)$
δ_{16}	16	w_1	$W(E_8)$	$W(E_8)$
$\bar{\delta}_{120}$	$120 \cdot 1$	w_2	$W(E_7A_1)$	$W(E_7A_1)$
$\bar{\delta}_{135}$	$135 \cdot 1$	$2w_1$	$W(D_8)$	$W(D_8)$

non-genuine fund. $\widetilde{K}$-type	$\#\delta_0$	$\#\bar{\delta}_{120}$	$\#\bar{\delta}_{135}$
w_2	0	1	0
w_4	35	7	7
w_6	28	35	28
w_8	8	1	0

genuine fund. $\widetilde{K}$-type	$\#\delta_{16}$
w_1	1
w_3	35
w_5	273
w_7	8

$\otimes$	δ_{16}	$\bar{\delta}_{120}$	$\bar{\delta}_{135}$
δ_{16}	$\delta_0 + \bar{\delta}_{120} + \bar{\delta}_{135}$	$120\delta_{16}$	$135\delta_{16}$
$\bar{\delta}_{120}$	$120\delta_{16}$	$120\delta_0 + 56\bar{\delta}_{120} + 56\bar{\delta}_{135}$	$63\bar{\delta}_{120} + 64\bar{\delta}_{135}$
$\bar{\delta}_{135}$	$135\,\delta_{16}$	$63\bar{\delta}_{120} + 64\bar{\delta}_{135}$	$135\delta_0 + 72\bar{\delta}_{120} + 70\bar{\delta}_{135}$

The two-fold cover of F_4

- $\widetilde{G} = \widetilde{F}_4$
- $\widetilde{K} = Sp(1) \times Sp(3)$

W-orbit of $\widetilde{M}$-types	dim.	fine $\widetilde{K}$-type	W_δ^0	$W(\delta)$
δ_0	1	$(0\|000)$	$W(F_4)$	$W(F_4)$
δ_2	2	$(1\|000)$	$W(F_4)$	$W(F_4)$
$\bar{\delta}_3$	$3 \cdot 1$	$(2\|000)$	$W(C_4)$	$W(C_4)$
$\bar{\delta}_6$	$3 \cdot 2$	$(0\|100)$	$W(B_4)$	$W(B_4)$
$\bar{\delta}_{12}$	$12 \cdot 1$	$(1\|100)$	$W(B_3A_1)$	$W(B_3A_1)$

non-genuine fund. $\widetilde{K}$-types	$\#\delta_0$	$\#\bar{\delta}_3$	$\#\bar{\delta}_{12}$
(0|000)	1	0	0
(0|110)	2	0	1

genuine fund. $\widetilde{K}$-types	$\#\delta_2$	$\#\bar{\delta}_6$
(1|000)	1	0
(0|100)	0	1
(0|111)	4	1

$\otimes$	δ_2	$\bar{\delta}_3$	$\bar{\delta}_6$	$\bar{\delta}_{12}$
δ_2	$\delta_0 + \bar{\delta}_3$	$3\delta_2$	$\bar{\delta}_{12}$	$4\bar{\delta}_6$
$\bar{\delta}_3$	$3\delta_2$	$3\delta_0 + 2\bar{\delta}_3$	$3\bar{\delta}_6$	$3\bar{\delta}_{12}$
$\bar{\delta}_6$	$\bar{\delta}_{12}$	$3\bar{\delta}_6$	$3\delta_0 + 3\bar{\delta}_3 + 2\bar{\delta}_{12}$	$12\delta_2 + 8\bar{\delta}_6$
$\bar{\delta}_{12}$	$4\,\bar{\delta}_6$	$3\bar{\delta}_{12}$	$12\delta_2 + 8\bar{\delta}_6$	$12\delta_0 + 12\bar{\delta}_3 + 8\bar{\delta}_{12}$

The two-fold cover of E_7

- $\widetilde{G} = \widetilde{E}_7$
- $\widetilde{K} = SU(8)$

W-orbit of $\widetilde{M}$-types	dim.	fine $\widetilde{K}$-type	W^0_δ	$W(\delta)$
δ_1	1	(0)	$W(E_7)$	$W(E_7)$
δ_8	8	w_1	$W(E_7)$	$W(E_7)$
$\delta_8^\star$	8	w_7	$W(E_7)$	$W(E_7)$
$\bar{\delta}_{28}$	$28 \cdot 1$	$w_2,\ w_6$	$W(E_6)$	$W(E_6) \ltimes \mathbb{Z}_2$
$\bar{\delta}_{36}$	$36 \cdot 1$	$2w_1,\ 2w_7$	$W(A_7)$	$W(A_7) \ltimes \mathbb{Z}_2$
$\bar{\delta}_{63}$	$63 \cdot 1$	$w_1 + w_7$	$W(D_6A_1)$	$W(D_6A_1)$

fundamental $\widetilde{K}$-types	$\#\delta_1$	$\#\bar{\delta}_{28}$	$\#\bar{\delta}_{36}$	$\#\bar{\delta}_{63}$	$\#\delta_8$	$\#\delta_8^\star$
w_0	1	0	0	0	0	0
w_1	0	0	0	0	1	0
w_2	0	1	0	0	0	0
w_3	0	0	0	0	0	7
w_4	7	0	0	1	0	0
w_5	0	0	0	0	7	0
w_6	0	1	0	0	0	0
w_7	0	0	0	0	0	1

$\otimes$	$\boldsymbol{\delta_8}$	$\boldsymbol{\delta_8^\star}$	$\boldsymbol{\bar{\delta}_{28}}$
$\boldsymbol{\delta_8}$	$\bar{\delta}_{28} + \bar{\delta}_{36}$	$\delta_1 + \bar{\delta}_{63}$	$28\delta_8^\star$
$\boldsymbol{\delta_8^\star}$	$\delta_1 + \bar{\delta}_{63}$	$\bar{\delta}_{28} + \bar{\delta}_{36}$	$28\delta_8$
$\boldsymbol{\bar{\delta}_{28}}$	$28\delta_8^\star$	$28\delta_8$	$28\delta_1 + 12\bar{\delta}_{63}$
$\boldsymbol{\bar{\delta}_{36}}$	$36\delta_8^\star$	$36\delta_8$	$16\bar{\delta}_{63}$
$\boldsymbol{\bar{\delta}_{63}}$	$63\delta_8$	$63\delta_8^\star$	$27\bar{\delta}_{28} + 28\bar{\delta}_{36}$

$\otimes$	$\boldsymbol{\bar{\delta}_{36}}$	$\boldsymbol{\bar{\delta}_{63}}$
$\boldsymbol{\delta_8}$	$36\delta_8^\star$	$63\delta_8$
$\boldsymbol{\delta_8^\star}$	$36\delta_8$	$63\delta_8^\star$
$\boldsymbol{\bar{\delta}_{28}}$	$16\bar{\delta}_{63}$	$27\bar{\delta}_{28} + 28\bar{\delta}_{36}$
$\boldsymbol{\bar{\delta}_{36}}$	$36\delta_1 + 20\bar{\delta}_{63}$	$36\bar{\delta}_{28} + 35\bar{\delta}_{36}$
$\boldsymbol{\bar{\delta}_{63}}$	$36\bar{\delta}_{28} + 35\bar{\delta}_{36}$	$63\delta_1 + 62\bar{\delta}_{63}$

Restriction to $\widetilde{M}$ of the fundamental $\widetilde{K}$-types

the example of $\widetilde{E}_6$

$\widetilde{G} = \widetilde{E}_6$
$\widetilde{K} = Sp(4)$
Fundamental $\widetilde{K}$-types: w_1, w_2, w_3, w_4
W-orbits of $\widetilde{M}$-types: δ_1, δ_8, $\bar{\delta}_{27}$, and $\bar{\delta}_{36}$

- $\mathrm{Res}_{\widetilde{M}}\, w_1 = \delta_8$, and $\mathrm{Res}_{\widetilde{M}}\, w_2 = \delta_{27}$ (*fine* $\widetilde{K}$*-types*)
- w_3 is genuine, and has dimension 48, hence $\mathrm{Res}(w_3) = 6\delta_8$
- $(w_4)^{\widetilde{M}}$ is the reflection repr. 6_p, because w_4 is the repr. of $\widetilde{K}$ on $\mathfrak{p}$. For dimensional reasons, $\mathrm{Res}(w_4) = 6\delta_1 \oplus \bar{\delta}_{36}$.

Tensor product of W-orbits of $\widetilde{M}$-types

some examples for $\widetilde{E}_6$

- $\delta_8 \otimes \delta_8 = \mathrm{Res}_{\widetilde{M}}[w_1 \otimes w_1] = \mathrm{Res}_{\widetilde{M}}[(0) \oplus w_2 \oplus 2w_1] = \delta_1 \oplus \bar{\delta}_{27} \oplus \bar{\delta}_{36}$
- $\bar{\delta}_{36} \otimes \bar{\delta}_{36} = \mathrm{Res}_{\widetilde{M}}[(2w_1) \otimes (2w_1)] =$

$$= \mathrm{Res}_{\widetilde{M}} \underbrace{[(0) \oplus w_2 \oplus (2w_1)]}_{fine \rightarrow \delta_0 \oplus \bar{\delta}_{27} \oplus \bar{\delta}_{36}} \oplus \mathrm{Res}_{\widetilde{M}} \underbrace{[(2w_2) \oplus (2w_1 + w_2) \oplus (4w_1)]}_{\text{“}new'' \rightarrow \mathrm{Res}=?}$$

First, we compute $(2w_2)^{\widetilde{M}}$. Because $(2w_2) \hookrightarrow (w_2 \otimes w_2)$ and

$$(w_2 \otimes w_2)^{\widetilde{M}} = \mathrm{Ind}_{W(\delta_{27})}^{W(E_6)} \mathrm{Hom}_{\widetilde{M}}(\delta_{27}, w_2) = \mathrm{Ind}_{W(D_5)}^{W(E_6)}(5|0)$$

we can write:

$$(2w_2)^{\widetilde{M}} = \underbrace{(w_2 \otimes w_2)^{\widetilde{M}}}_{1_p \oplus 6_p \oplus 20_p} - \underbrace{(w_1 + w_3)^{\widetilde{M}}}_{\varnothing} - \underbrace{w_4^{\widetilde{M}}}_{6_p} - \underbrace{0^{\widetilde{M}}}_{1_p} = 20_p.$$

Similarly, we find $(4w_1)^{\widetilde{M}} = 15_q$. Then

$$\mathrm{Res}_{\widetilde{M}}(4w_1) = 15\delta_1 \oplus b\bar{\delta}_{27} \oplus c\bar{\delta}_{36}.$$

Comparing dimensions, we find that $35 = 3b + 4c$ hence $c = 2$, 5 or 8. We also notice that $c = \dim[\mathrm{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1)]$. Because

$$\mathrm{Ind}_{W(A_5A_1)}^{W(E_6)} \mathrm{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1) = (2w_1 \otimes 4w_1)^{\widetilde{M}} \supseteq (4w_1)^{\widetilde{M}} = 15_q$$

the $W(A_5A_1)$-representation $\mathrm{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1)$ is a submodule of

$$\mathrm{Res}_{W(A_5A_1)}^{W(E_6)}[15_q] = \underbrace{[(33) \otimes (11)]}_{\text{dim .5}} \oplus \underbrace{[(42) \otimes (2)]}_{\text{dim .9}} \oplus \underbrace{[(6) \otimes (2)]}_{\text{dim .1}}.$$

Hence $c = 5$, and $\mathrm{Res}_{\widetilde{M}}(4w_1) = 15\delta_1 \oplus 5\bar{\delta}_{27} \oplus 5\bar{\delta}_{36}$.
The restrictions of $(2w_1 + w_2)$ and $(2w_2)$ are computed similarly.
Then

$$\bar{\delta}_{36} \otimes \bar{\delta}_{36} = 36\delta_1 \oplus 20\bar{\delta}_{27} \oplus 20\bar{\delta}_{36}.$$

PART 6

- Standard Notation
- Multiplicities of K-types in principal series
- Some easy examples (*linear case*)
- Non-linear case (*what we know...*)
- An inductive algorithm to compute multiplicities
- **Generalization**

An inductive algorithm to compute multiplicities (revisited)

INPUT

tensor product of orbits of $\widetilde{M}$-types

restriction to $\widetilde{M}$ of fundamental $\widetilde{K}$-types

OUTPUT

dimension of $\mathrm{Hom}_{\widetilde{M}}(\delta, \mu)$

$\forall\ \widetilde{M}$**-type** δ **and** $\forall\ \widetilde{K}$**-type** μ

Generalization

INPUT

tensor product of orbits of $\widetilde{M}$-types

restriction to $\widetilde{M}$ of fundamental $\widetilde{K}$-types

$\mathrm{Hom}_{\widetilde{M}}(\delta, w \otimes \mu_\tau)$
$\forall$ **fund.** $\widetilde{K}$**-type** w **and** $\forall$ $\widetilde{M}$**-type** δ, τ

double stabilizer $W(\delta, \tau)$

OUTPUT

$\mathrm{Hom}_{\widetilde{M}}(\delta, \mu)$

$\forall$ $\widetilde{M}$**-type** δ **and** $\forall$ $\widetilde{K}$**-type** μ

$\uparrow$

as a $W(\delta)$*-representation*

DETAILS

... coming soon...