Suggested References (← none of them is mandatory)

1. Linear Representations of finite groups
   by Jean-Pierre Serre

2. Representation Theory. A first course
   by W. Fulton, J. Harris

3. Representations and characters of groups
   by G. James and M. Liebeck

4. Representations of finite and compact groups
   by Barry Simon

5. Character Theory of finite groups
   by I. Martin Isaacs

6. The symmetric group
   by Bruce E. Sagan

7. Young Tableaux
   by W. Fulton

Lecture notes will be provided...

LECTURE 1

1.1 Linear representations

1.2 Similarity of representations

1.3. Faithful representations

1.4 Direct sum of representations

1.5 Tensor product of representations

1.6 Dual representation

1.1 Linear Representations

Let G be a finite group. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $GL(V)$ be the group of isomorphism of V into itself. [By fixing a basis of V, we can identify $GL(V)$ with the group $GL(n,\mathbb{C})$ of $n \times n$ invertible matrices $/_{\mathbb{C}}$, with $n = \dim V$].

▶ **Def.** A (linear) representation of G in V is a group homomorphism $\rho : G \to GL(V)$.

[To define a representation of G in V, we associate with each element s of G, an authomorphism $\rho(s)$ of V so that:

- $\rho(ss') = \rho(s)\rho(s') \quad \forall s, s' \in G$
- $\rho(1) = \mathbb{1}_V$
- $\rho(s^{-1}) = \rho(s)^{-1} \quad \forall s \in G$.]

▶ **Def-** The degree of a representation (ρ, V) is the dimension of the complex vector space V.

▶ **Remark-** Every group has a representation of arbitrarily large degree. Indeed, for all $n \geq 1$, we can consider the "Trivial representation" $\rho : G \to GL(\mathbb{C}^n) \simeq GL(n,\mathbb{C})$, $s \mapsto I_n$.

▶ Examples of linear representations

① [degree = 1]

A representation of G of degree 1 is a group homomorphism

$$\rho : G \mapsto GL(\mathbb{C}) \cong GL(1,\mathbb{C}) \cong \mathbb{C}^*$$

↳ multiplicative group of non-zero complex numbers

Because G is a finite group, every element s of G has finite order (say $s^n = 1$); Then every value $\rho(s)$ is a root of unity ($1 = \rho(s^n) = \rho(s)^n$).

complex #s of $|\cdot| = 1$

In particular, The image of ρ is a finite subgroup of S^1.

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② [degree = $|G|$]

Let n be The order of the group G. Let us construct a complex vector space V of dimension n.

For each element $g \in G$, consider a symbol $\underline{e}_g$. Define V To be The set of formal linear combinations of such symbols.

$$V = \{ \sum_{g\in G} a_g \underline{e}_g : a_g \in \mathbb{C} \}.$$

We define a representation of G on V as follows: if $s \in G$, $\rho(s)$ is The authomorphism of V which acts on The basis $\{\underline{e}_g\}$ by : $\rho(s)\underline{e}_g = \underline{e}_{sg}$

It's easy To check That The correspondence

$$\rho : G \to GL(V), \; s \mapsto \rho(s)$$

is indeed a group homomorphism: $\rho(1)\underline{e}_g = \underline{e}_g$ and

- $\rho(s_1 s_2)\underline{e}_g = \underline{e}_{(s_1 s_2)g} = \underline{e}_{s_1(s_2 g)} = \rho(s_1)\underline{e}_{s_2 g} = \rho(s_1)\rho(s_2)\underline{e}_g$

$\forall s_1, s_2, g \in G$. We call ρ The "<u>regular representation</u>".

By construction, The regular representation of G has degree $|G|$.

③ [A generalization of the previous example].

Given any action of G on a finite set X, we can construct a representation of G (of degree $=|X|$) by letting

- $V = \{ \sum_{x\in X} a_x \underline{e}_x : x \in X \}$ (a vector space with basis $\mathcal{B} = \{\underline{e}_x\}_{x\in X}$)

and by defining

- $\rho(s)\underline{e}_x = \underline{e}_{s\cdot x}$.

We call ρ the "<u>permutation representation</u>" associated to the given action of G on X.

[the regular representation is the permutation representation associated with the action of G on itself by left multiplication.]

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④ [A representation of D_8].

Let G be the dihedral group of order 8:

$$G = \langle a, b : a^4 = 1,\ b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

[Every element of G can be written uniquely in the form $s = a^i b^j$ with $i = 0,1,2,3$ and $j = 0,1$.].

Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Because these matrices satisfy the relations

$$A^4 = I,\quad B^2 = I,\quad B^{-1}AB = A^{-1},$$

we obtain a well defined group homomorphism:

$$\rho : D_8 \mapsto GL(2,\mathbb{C}),\quad a^i b^j \longmapsto A^i B^j.$$

ρ is a representation of D_8 of degree 2.

$\boxed{1.2}$ Similarity of representations

Let (ρ, V) and (ρ', V') be representations of the same group G in vector spaces V and V'.

▶ <u>Def</u> - We say that (ρ, V) is "<u>equivalent</u>" (or "<u>similar</u>" or "<u>isomorphic</u>") to (ρ', V') if there exists a linear isomorphism $T: V \to V'$ that makes the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho(s)} & V \\ T\downarrow & & \downarrow T \\ V' & \xrightarrow{\rho'(s)} & V' \end{array}$$

commutative, for all s in G.
In other words, the isomorphism $T: V \to V'$ should satisfy the condition: $T \circ \rho(s) = \rho'(s) \circ T, \ \forall s \in G$.
[We call T an "intertwining operator" between the two representations.]

▶ (Obvious) <u>remarks</u>: $\boxed{A}$. Similar representations have the same degree. The reverse is not necessarily true.
$\boxed{B}$. Similarity of representations is an equivalence relation.

▶ <u>Examples of similar representations</u>

① Let ρ be the trivial representation of degree n. Then ρ is the only representation which is equivalent to ρ.

② Let W be a finite dimensional vector space, and let $\tau : G \to GL(W)$ be a representation of G in W with the following property: there exists an element $w_0 \in W$ s.t. the set $\mathcal{B} = \{ \tau(g) w_0 : g \in G \}$ is a basis of W. Then τ is equivalent to the regular representation.

[Proof: Let's verify that the linear isomorphism

$$T : W = \mathrm{Span}(\tau(g) w_0)_{g \in G} \longrightarrow V = \mathrm{Span}(\underline{e}_g)_{g \in G}$$

$$\tau(g) w_0 \longmapsto \underline{e}_g$$

is an intertwining operator between (τ, W) and the regular representation (ρ, V), i.e. let's show that ~~To show:~~ the diagram

$$\begin{array}{ccc} W & \xrightarrow{\tau(s)} & W \\ T \downarrow & & \downarrow T \\ V & \xrightarrow{\rho(s)} & V \end{array}$$

is commutative, for all s in G.

$$T(\tau(s) \underbrace{\tau(g) w_0}_{\text{a basis element}}) = T(\tau(sg) w_0) = \underline{e}_{sg} =$$

$$= \rho(s) \underline{e}_g = \rho(s) T(\tau(g) w_0). \checkmark \;]$$

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③ Consider the representation

$$\rho : D_8 \mapsto GL(2, \mathbb{C}), \; a^i b^j \mapsto A^i B^j$$

constructed before $\left(A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right.$ and $\left.B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$, and let $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

It's easy to check that the matrices

$$\tilde{A} = C^{-1}AC \quad \text{and} \quad \tilde{B} = C^{-1}BC$$

also satisfy the conditions:

$$\tilde{A}^4 = I \; ; \; \tilde{B}^2 = I \; ; \; \tilde{B}^{-1}\tilde{A}\tilde{B} = \tilde{A}^{-1}.$$

So we obtain a new representation $\tilde{\rho} : D_8 \longrightarrow GL(2, \mathbb{C})$,
$a^i b^j \longmapsto \tilde{A}^i \tilde{B}^j$

which is equivalent to ρ.

An intertwining operator between $(\rho, \mathbb{C}^2)$ and $(\tilde{\rho}, \mathbb{C}^2)$ is given by $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $\underline{v} \longmapsto C^{-1}\underline{v}$.

Indeed the diagram

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "A^i B^j"] \arrow[d, "C^{-1}"'] & \mathbb{C}^2 \arrow[d, "C^{-1}"] \\
\mathbb{C}^2 \arrow[r, "\tilde{A}^i \tilde{B}^j"] & \mathbb{C}^2
\end{tikzcd}$$

$$\begin{array}{c} \| \\ C^{-1}(A^i B^j)C \end{array}$$

is commutative for all $i = 0, \dots, 3$ and all $j = 1, 2$. ∎

1.3 Faithful Representations

Let ρ be a representation of G in V.
We define the kernel of ρ to be the set

$$\operatorname{Ker}\rho = \{ s \in G : \rho(s) = \mathbb{1}_V \}.$$

$\Rightarrow \operatorname{Ker}\rho$ is a normal subgroup of G, and $\operatorname{Ker}\rho = G$ when ρ is the trivial representation.

▸ **Def** - A representation (ρ, V) of G in V is called "*faithful*" if the kernel of ρ is trivial : $\operatorname{Ker}\rho = \{1\}$.

Clearly ρ is faithful if and only if $\operatorname{Im}\rho$ is isomorphic to G. So if G has a faithful representation, then we can realize G as a finite subgroup of $GL(n, \mathbb{C})$ for some n.

▸ **Remark** - Every finite group has a faithful representation. Indeed, the regular representation is always faithful.

$[\ \rho(s) = \mathbb{1}_V \iff \rho(s) e_g = e_g \ \forall g \in G \iff e_{sg} = e_g \ \forall g \in G$

$\iff sg = g \ \forall g \in G \iff s = 1\ ]$.

▸ **Examples / Non examples**

① The trivial representation of degree n is faithful if and only if $|G| = 1$, ie. $G = \{1\}$.

② Let $G = C_4$ be the cyclic group of order 4, and let a be a generator.

The representation $\rho : G \to GL(2, \mathbb{C})$, $a \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is not faithful ($\leftarrow \rho(a^2) = I_2$), but

$\rho' : G \to GL(2, \mathbb{C})$, $a \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is faithful.

1.4 Direct sum of representations

Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G.
We define a representation $\rho = \rho_1 \oplus \rho_2$ of G in the vector space $V = V_1 \oplus V_2$ (called "direct sum of the representations ρ_1 and ρ_2").
For all s in G, v_1 in V_1, v_2 in V_2:

$$\rho(s)(v_1, v_2) \overset{\text{def}}{=} (\rho_1(s)v_1, \rho_2(s)v_2).$$

Because $\rho_1(s) \in GL(V_1)$, $\rho_2(s) \in GL(V_2)$, we have $\rho(s) \in GL(V_1 \oplus V_2)$.
The mapping $\rho : G \to GL(V_1 \oplus V_2)$, $s \mapsto \rho(s)$ is a well defined representation. Indeed: $\rho(1) = \mathbb{1}_{V_1 \oplus V_2}$, and

$$\rho(ts)(v_1, v_2) = (\rho_1(ts)v_1, \rho_2(ts)v_2) =$$
$$= (\rho_1(t)\rho_1(s)v_1, \rho_2(t)\rho_2(s)v_2) = \rho(t)(\rho_1(s)v_1, \rho_2(s)v_2) =$$
$$= \rho(t)\rho(s)(v_1, v_2) \qquad \forall s, t \in G;\ v_1 \in V_1;\ v_2 \in V_2.$$

▶ Remark - We can choose a basis of $V_1 \oplus V_2$ of the form

$$(\underbrace{v_1^1, \dots, v_1^m}_{\mathcal{B}_1 = \text{basis of } V_1}, \underbrace{v_2^1, \dots, v_2^n}_{\mathcal{B}_2 = \text{basis of } V_2}).$$

The matrix of $\rho(s)$ w.r.t. this basis is a block matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.
with A_i = the matrix of $\rho_i(s)$ w.r.t. the basis $\mathcal{B}_i$ $(i = 1, 2)$.

1.5 Tensor product of representations

Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G.
We define a representation $\rho = \rho_1 \otimes \rho_2$ of G on the vector space $V = V_1 \otimes V_2$ (called "the tensor product of ρ_1 and ρ_2").

Symbols of the form $v_1 \otimes v_2$, with $v_1 \in V_1$ and $v_2 \in V_2$ form a generating system for $V_1 \otimes V_2$. We define

$$\rho(s)(v_1 \otimes v_2) = \rho(s) v_1 \otimes \rho(s) v_2 \qquad \forall v_1 \otimes v_2 \in V_1 \otimes V_2$$

and extend $\rho(s)$ by linearity to all $V_1 \otimes V_2$.

The mapping $\rho: G \longrightarrow GL(V_1 \otimes V_2)$, $s \mapsto \rho(s)$ is a group homomorphism. Indeed, for all $s, t \in G$ and $v_1 \otimes v_2 \in V_1 \otimes V_2$ we have:

$$\rho(st)(v_1 \otimes v_2) = \rho_1(st) v_1 \otimes \rho_2(st) v_2 =$$

$$= [\rho_1(s) \rho_1(t) v_1] \otimes [\rho_2(s) \rho_2(t) v_2] =$$

$$= \rho(s) [\rho_1(t) v_1 \otimes \rho_2(t) v_2] = \rho(s) \rho(t) (v_1 \otimes v_2).$$

Moreover, $\rho(1) = \mathbb{1}_{V_1 \otimes V_2}$.

▲ Remark - Let $\{v_1^i\}_{i=1 \dots m} = \mathcal{B}_1$ be a basis of V_1, and $\{v_2^j\}_{j=1 \dots n} = \mathcal{B}_2$ be a basis of V_2. Then $\{v_1^i \otimes v_2^j\}_{\substack{i=1 \dots m \\ j=1 \dots n}} = \mathcal{B}$ is a basis of $V_1 \times V_2$.

Choose the ordering:

$$\boxed{v_1^1} \otimes v_2^1, \; \boxed{v_1^1} \otimes v_2^2, \dots, \boxed{v_1^1} \otimes v_2^n, \; v_1^2 \otimes v_2^1, \dots, v_1^2 \otimes v_2^n, \dots$$

Then, if A is the matrix of $\rho_1(s)$ wrt $\mathcal{B}_1$, and B is the matrix of $\rho_2(s)$ wrt $\mathcal{B}_2$,

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1m} B \\ a_{21} B & a_{22} B & \cdots & a_{2m} B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} B & a_{m2} B & \cdots & a_{mm} B \end{pmatrix}$$

is the matrix of $\rho(s) = \rho_1(s) \otimes \rho_2(s)$ wrt $\mathcal{B}$.

2.6 Dual Representation

Let ρ be a representation of G in a vector space V. We want to define a representation ρ^* of G on the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$, in such a way that the two representation of G respect the natural pairing $\langle , \rangle$ between V and V^*:

for all v in V, φ in V^*, $g \in G$, we need

$$\langle \rho^*(g)\varphi, \rho(g)v \rangle = \langle \varphi, v \rangle.$$

This requires:

$$(\rho^*(g)\varphi)(\rho(g)v) = \varphi(v) \quad \forall v \in V.$$

Choose $v = \rho(g^{-1})v'$

$$\Rightarrow (\rho^*(g)\varphi)(v') = \varphi(\rho(g^{-1})v') \qquad \forall v' \in V.$$

So $\rho^*(g)$ acts on $V^* = \mathrm{Hom}(V, \mathbb{C})$ by "$\varphi \mapsto \varphi \circ \rho(g^{-1})$".
<u>It</u>'s easy to check that the map

$$\rho^* : G \longrightarrow GL(V^*), \; g \mapsto \rho^*(g)$$

is well defined, and that ρ^* is a group homomorphism:

- $\rho^*(g_1 g_2)(\varphi)(v) = \varphi(\rho(g_1 g_2)^{-1} v) = \varphi(\rho(g_2^{-1} g_1^{-1})v) =$

$$= \varphi(\rho(g_2^{-1})\rho(g_1^{-1})v) = \rho(g_2)\varphi(\rho(g_1^{-1})v) =$$

$$= \rho(g_1)\rho(g_2)\varphi(v) \quad \forall v \in V, \varphi \in V^*, g_1, g_2 \in G.$$

- $\rho^*(1) = \mathbb{1}_{V^*}$.

Remark – Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V, denote by $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$ the corresponding dual basis of V^* (here $v_i^* : V \to \mathbb{C}$, $v_j \mapsto \delta_{ij}$).

If A_s is the matrix of $\rho(s)$ wrt $\mathcal{B}$, the matrix of $\rho^*(s)$ wrt $\mathcal{B}^*$ is $(A_s^{-1})^T$.

Let's prove it. For simplicity of notations, let $B = A_s^{-1}$.

Then for all $i, j = 1 \ldots n$ we can write:

$$\rho^*(s)(v_j^*)(v_i) = v_j^*(\rho(s^{-1}) v_i) = v_j^*\left(\sum_{k=1}^n b_{ki} v_k\right) =$$

$$= b_{ji} = \left[\sum_{l=1}^n b_{jl} v_l^*\right](v_i)$$

$$\Rightarrow \quad \rho^*(s)(v_j^*) = \sum_{l=1}^n b_{jl} v_l^*$$

$$\Rightarrow \quad \rho^*(s) \underset{\text{wrT } \mathcal{B}^*}{\rightsquigarrow} B^T = (A_s^{-1})^T.$$

Summary of Lecture 1

Let G be a finite group, and let V be a vector space over $\mathbb{C}$.

1. A representation of G in V is a group homomorphism $G \to GL(V)$
2. The degree of a representation (ρ, V) is equal to the dim. of V
3. Two representations (ρ_1, V_1), (ρ_2, V_2) of G are equivalent if and only if there exist an isomorphism $T: V_1 \to V_2$ s.T.

$$T \circ \rho_1(s) = \rho_2(s) \circ T \quad \forall s \in G$$

4. A representation ρ is faithful if $\operatorname{Ker}\rho = \{1\}$
5. The direct sum of Two representations $(\rho_1, V_1), (\rho_2, V_2)$ of G is the representation of G in $V_1 \oplus V_2$ defined by

$$\rho(s)(v_1, v_2) = (\rho(s) v_1, \rho(s) v_2).$$

You can find a basis $\mathcal{B}_1$ of V_1, a basis $\mathcal{B}_2$ of V_2, and a basis $\mathcal{B}$ of $V_1 \oplus V_2$ sT

$$\rho_1(s) \rightsquigarrow A_1$$
$$\rho_2(s) \rightsquigarrow A_2$$
$$\rho(s) \rightsquigarrow \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}.$$

6. The direct product of Two representations (ρ_1, V_1) and (ρ_2, V_2) of G is the representation of G in $V_1 \otimes V_2$ defined by

$$\rho(s)(v_1 \otimes v_2) = \rho_1(s) v_1 \otimes \rho_2(s) v_2.$$

You can find bases $\mathcal{B}_i$ of V_i $(i = 1, 2)$, and a basis $\mathcal{B}$ of $V_1 \otimes V_2$,

s.T. $\rho_1(s) \rightsquigarrow A$
$\rho_2(s) \rightsquigarrow B$

$$\rho(s) \rightsquigarrow A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

7. The dual of (ρ, V) is the representation ρ^* of G in $V^* = \operatorname{Hom}(G, \mathbb{C})$ defined by $\rho^*_{(s)}\varphi(v) = \varphi(\rho(s)^{-1} v)$.
8. Every group has a faithful representation, for instance the regular repr.

Suggested problems for Lecture 1

1. Let G be a cyclic group of order m, with generator a. Fix $A \in GL(n,\mathbb{C})$.
Define $\rho : G \to GL(n,\mathbb{C})$, $a^r \mapsto A^r$.
Explain when ρ is a representation of G, and when it is a faithful representation.

2. Let G be the dihedral group $D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.
Set $A = \begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{pmatrix}$; $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(i) Prove that $\rho : D_{12} \to GL(2,\mathbb{C})$, $a^r b^s \mapsto A^{3r}(-B)^s$ is a well defined representation.
(ii) Decide whether ρ is faithful.

3. Let W be the space of all complex valued functions on G.
Define a representation μ of G on W by
$(\mu(s)\alpha)(t) = \alpha(s^{-1}t)$ $\forall t, s \in G$, $\forall \alpha \in W$.
Show that (μ, W) is isomorphic to the regular representation.

4. Show that for all representations U, V, W of G

$$V \otimes (U \oplus W) \cong (V \otimes U) \oplus (V \otimes W).$$

5. If $(\rho_1, V_1), (\rho_2, V_2)$ are representations of G, show that the mapping

$$\rho : G \longrightarrow GL(\mathrm{Hom}(V_1, V_2))$$
$$g \longmapsto \rho(g)$$

defined by

$$(\rho(g)\varphi)(v_1) = \rho_2(g)\,\varphi(\rho_1(g^{-1})v_1) \quad \forall g \in G,\ \varphi \in \mathrm{Hom}(V_1, V_2),\ \forall v_1 \in V_1$$

is a representation of G.

6. Show that, as a representation of G, $\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$.