$\boxed{\text{LAST TIME}}$

- $x = \{x_1, x_2, \dots\}$ infinite set of variables
- $n \geq 0$, $\alpha = (\alpha_1, \alpha_2, \dots)$ composition of n $\Rightarrow$ $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ $\boxed{\text{monomial of degree } n}$
- $\Lambda^n = \{ f = \sum_{\alpha:\text{ comp. of } n} c_\alpha x^\alpha : f \text{ symmetric} \}$

$$f(x_1 x_2 \dots) = f(x_{\pi(1)} x_{\pi(2)} \dots) \quad \forall \pi \text{ permut. of } \mathbb{Z}_{\geq 1}$$

- $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$

We are interested in finding basis of Λ

- <u>monomial symm functions</u>

$\lambda \vdash n \quad (\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots))$

$$m_\lambda = \sum_{\alpha = \text{distinct permutations of } \lambda} x^\alpha$$

Example : $n = 3$, $\lambda = (2,1) \Rightarrow m_{(2,1)} = \sum_{i,j} x_i^2 x_j$

$\lambda = (1^3) \Rightarrow m_{(1^3)} = \sum_{i<j<k} x_i x_j x_k$

$\Rightarrow \{m_\lambda\}_{\lambda \vdash n}$ basis of Λ^n

and $\{m_\lambda\}_{\text{all } \lambda}$ basis of Λ.

- elementary symmetric functions

- $\forall k \geq 1$, $e_k = \sum_{i_1<i_2<\dots<i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$

and $e_0 = 1$.

- $\forall \lambda \vdash n$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $e_\lambda \equiv e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$

example $n=3$, $\lambda = (2,1)$

$$e_\lambda = \left(\sum_{i<j} x_i x_j\right)\left(\sum_k x_k\right)$$

$\Rightarrow$ $\{e_\lambda\}_{\lambda \vdash n}$ is a new basis of Λ^n

$\Rightarrow$ $\{e_\lambda\}_{\text{all } \lambda}$ is a new basis of Λ

Change of basis (from m_μ's to e_λ's)

integers ≥ 0

$$e_\lambda = \underset{\substack{\uparrow \\ \text{dual partition}}}{m_{\lambda'}} + \underset{\substack{\uparrow \\ \text{dominance order}}}{\sum_{\mu \triangleleft \lambda'}} a_{\lambda'\mu} m_\mu$$

Notice that each e_λ is a l.c. of m_μ's with integer coefficients.

$\boxed{\text{TODAY}}$

We introduce new basis of Λ:

- h_λ = complete symmetric functions
- s_λ = Shur functions

(possibly the power sums also...)

Complete Homogeneous Symmetric functions

Set $h_0 = 1$ and

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \text{sum of all monomials of degree } n$$
$$= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ \uparrow \\ (\text{also equal})}} x_{i_1} x_{i_2} \dots x_{i_n},$$

for every $n \geq 1$. Example: $h_3 = \left(\sum_i x_i^3\right) + \left(\sum_{i,j} x_i^2 x_j\right) + \left(\sum_{i<j<e} x_i x_j x_e\right)$.
Also define $h_{\lambda=(\lambda_1 \dots \lambda_k)} = h_{\lambda_1} \cdot h_{\lambda_2} \cdot \dots \cdot h_{\lambda_k}$,
for every partition λ of n.

The complete symmetric functions are in many ways "dual" to the elementary symmetric functions. We will show that there exists an involution ω of Λ that carries e_n into h_n, $\forall n \geq 0$. Because ω preserves multiplication, ω also carries e_λ into h_λ for all partitions λ.

The easiest way to compare e_n and h_n is to compare the corresponding generating functions.

Lemma - ① $E(t) = \sum_{n \geq 0} e_n(x) t^n = \prod_{i \geq 1} (1 + x_i t)$

② $H(t) = \sum_{n \geq 0} h_n(x) t^n = \prod_{i \geq 1} \frac{1}{(1 - x_i t)}$.

proof – ① Consider the product $\prod_{i\geq 1}(1+x_i t)$, and expand it as a series of powers of t.
The coefficient of t^n is

$$\sum_{i_1<\dots<i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = e_n(x).$$

Hence we can write

$$\prod_{i\geq 1}(1+x_i t) = \sum_{n\geq 0} e_n(x)\, t^n \overset{\text{by def.}}{=} E(t).$$

② $$\prod_{i\geq 1} \frac{1}{1-x_i t} = \prod_{i\geq 1}\left[\sum_{n_i\geq 0} (x_i t)^{n_i}\right] = \prod_{i\geq 1}\left[\sum_{n_i\geq 0} x_i^{n_i} t^{n_i}\right].$$

Expand this product as a power series in t.
The coefficient of t^n is

$$\sum_{\alpha:\text{compositions of } n} x^\alpha = \text{sum of all monomials of degree } n = h_n(x).$$

$$\Rightarrow \prod_{i\geq 1} \frac{1}{1-x_i t} = \sum_{n\geq 0} h_n(x)\, t^n \overset{\text{by def.}}{=} H(t). \quad \square$$

Corollary $H(t)\,E(-t) = 1$. $\quad \left[\, H(t)\,E(-t) = \left[\prod_i \frac{1}{(1-x_i t)}\right]\left[\prod_i (1-x_i t)\right] = 1 \,\right]$

The proof of this corollary is trivial, but its consequences are big !!!
Indeed, we obtain:

$$1 = H(t)\,E(-t) = \left[\sum_{a\geq 0} h_a(x)\, t^a\right]\left[\sum_{b\geq 0} e_b(x)\,(-1)^b\, t^b\right] =$$

$$= \sum_{n \geq 0} \left[\sum_{a+b=n} (-1)^b \, e_b(x) \, h_a(x) \right] t^n$$

$$\Longrightarrow \quad \text{For all } n \geq 1, \quad \boxed{\sum_{a+b=n} (-1)^b \, e_b(x) \, h_a(x) = 0}$$

Equivalently:

$$\sum_{a=0 \ldots n} (-1)^{n-a} \, h_a(x) \, e_{n-a}(x) = 0, \quad \forall n \geq 1.$$

Example:

$$n=1 \Rightarrow \quad h_1(x) \underbrace{e_0(x)}_{1} - \underbrace{h_0(x)}_{1} e_1(x) = 0 \quad \Rightarrow h_1(x) = e_1(x)$$

$$n=2 \Rightarrow \quad h_2(x) \underbrace{e_0(x)}_{1} - \underbrace{h_1(x)}_{e_1(x)} e_1(x) + \underbrace{h_0(x)}_{1} e_2(x) = 0$$

$$\Rightarrow \quad h_2(x) = [e_1(x)]^2 - e_2(x) = e_{(1^2)} - e_{(2)}$$

$$n=3 \Rightarrow \quad h_3(x) \underbrace{e_0(x)}_{1} - \underbrace{h_2(x)}_{[e_1(x)]^2 - e_2(x)} e_1(x) + \underbrace{h_1(x)}_{e_1(x)} e_2(x) - \underbrace{h_0(x)}_{1} e_3(x) =$$

$$\Rightarrow \quad h_3(x) = e_1(x)^3 - 2e_2(x) e_1(x) + e_3(x) = e_{(1^3)} - 2e_{(21)} + e_{(3)}$$

You can express $h_n(x)$ as a linear combination (with integer coefficient) of e_λ's, with $\lambda \vdash n$.

Define a map

$$\omega : \Lambda \longrightarrow \Lambda, \quad e_r \longrightarrow h_r .$$

the set $\{e_r\}_{r\geq 0}$ forms a basis of Λ as a ring (in the sense that every element of Λ can be written uniquely a polynomial in the e_r's).

So the map $e_r \to h_r$ extends uniquely to an endomorphism $\omega : \Lambda \to \Lambda$.

this endomorphism respects multiplication and carries e_λ into h_λ, for all $\lambda \vdash n$ and all n.

<u>Claim</u> - ω is an involution, i.e. $\omega^2 =$ identity.

<u>proof</u> - we must show that $\omega(h_r) = e_r$ $\forall r \geq 0$, but this is easy because the e_r's and h_r's satisfy a relation

$$\sum_{a+b=n} (-1)^b \, h_a(x) \, e_b(x) = 0 \qquad (*)$$

which is perfectly symmetric in h and e.

Apply ω to $(*)$: $\boxed{\forall n \geq 1}$

$$\sum_{a+b=n} (-1)^b \, \omega(h_a(x)) \, h_b(x) = 0 \qquad (\boxdot)$$

We show that $\omega(h_n) = e_n$ by induction on n.

If $n = 0$, then $\omega(h_0(x)) = \omega(1) = \omega(e_0) = h_0 = 1 = e_0(x)$.

If $n > 0$, then $(\boxdot)$ gives:

$$\omega(h_n(x)) \underbrace{h_0(x)} = - \sum_{\substack{a+b=n \\ a \neq n}} (-1)^b \underbrace{\omega(h_a(x))}_{= e_a(x) \text{ by induction}} h_b(x) =$$

$$= -\sum_{\substack{a+b=n \\ a\neq n}} (-1)^b e_a(x) h_b(x) \underset{(*)}{=} e_n(x) \underbrace{h_0(x)}_{1} = e_n(x)$$

So $w^2 = \mathbb{1}$, i.e. w is an involution. □
[By construction, w carries e_λ into h_λ, and h_λ into e_λ].

Corollary - The set $\{h_\lambda : \underset{\lambda \geq 0}{\lambda \vdash n}\}$ is a basis for Λ.

proof - $\{h_\lambda\}$ spans Λ: if $f \in \Lambda$, then we can write

$$f \underset{g = w(f)}{=} w(g) = w\left(\sum_\lambda c_\lambda e_\lambda\right) = \sum_\lambda c_\lambda w(e_\lambda) = \sum_\lambda c_\lambda h_\lambda .$$

Moreover, the h_λ's are l.i. : $0 = \sum a_\lambda h_\lambda \underset{\text{apply } w}{\Leftrightarrow} 0 = \sum_\lambda a_\lambda w(h_\lambda)$

$\Leftrightarrow 0 = \sum_\lambda a_\lambda e_\lambda \Leftrightarrow a_\lambda = 0, \forall \lambda$. □

Remark - Define $\Lambda_{\mathbb{Z}} = \bigoplus_{n \geq 0} \Lambda^n_{\mathbb{Z}}$, and

$\Lambda^n_{\mathbb{Z}}$ = n-homogeneous symmetric formal power series with integer coefficients.

then

- $\{m_\lambda\}$ is a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{Z}}$
- $\{e_\lambda\}$ is a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{Z}}$
- $\{h_\lambda\}$ is also a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{Z}}$.

We conclude our analysis of the complete homogeneous functions with some lemmas that we'll be useful later.

<u>Lemma</u> For every partition μ of n, write

$$h_\mu = \sum_\rho c_{\mu\rho} m_\rho .$$

Then $c_{\mu\rho}$ is the number of $\{0,1\}$-matrices with row-sums ρ and column-sums μ.

<u>proof</u> - Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be any composition obtained by permuting the entries of $\rho = (\rho_1, \rho_2, \dots)$. Because h_μ is symmetric, the coefficient of x^α in h_μ equals the coefficient $c_{\mu\rho}$ of m_ρ in h_μ. We want to compute this coefficient.

Because

$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_r} = \Big(\sum_{i_1 < \dots < i_{\mu_1}} x_{i_1} \cdots x_{i_{\mu_1}}\Big)\Big(\sum_{j_1 < \dots < j_{\mu_2}} x_{j_1} \cdots x_{j_{\mu_2}}\Big) \cdots$$

$$\cdots \Big(\sum_{l_1 < \dots < l_{\mu_r}} x_{l_1} \cdots x_{l_{\mu_r}}\Big),$$

the coefficient of $x^\alpha = x^{\alpha_1} x^{\alpha_2} \cdots$ in h_μ equals the number of ways to write x^α as a product

$$x^\alpha = (x_{i_1} \cdots x_{i_{\mu_1}})(x_{j_1} \cdots x_{j_{\mu_2}}) \cdots (x_{l_1} \cdots x_{l_{\mu_r}}) \qquad (*)$$

with $i_1 < \dots < i_{\mu_1}, \dots, l_1 < \dots < l_{\mu_r}$.

[+]

For each choice e of indices satisfying ⊛ we constr[uct] a matrix A_e with entries $\{0,1\}$ as follows:

- The first column of A_e has a 1 in positions $i_1, \dots, i_{\mu_1}$ and a zero everywhere else
- the second column of A_e has a 1 in positions $j_1 \cdots j_{\mu_2}$, and a 0 everywhere else

⋮

- The r^{Th} column of A_e has a 1 in position $l_1 \cdots l_{\mu_r}$ and a zero everywhere else
- the remaining columns of A_e are identically zero.

This matrix has infinitely many columns and infinitely many rows. We notice that

- The sum of the entries of the various columns are

$$\mu_1, \mu_2, \dots, \mu_r$$

and

- The sum of the entries of the various rows are

$$\alpha_1, \alpha_2, \dots .$$

The mapping $e \longrightarrow A_e$ is a bijection between the set of monomials satisfying (⊛) and the set of $\{0,1\}$ matrices with column-sums μ and row-sums α.

$$\Rightarrow c_{\mu\rho} = c_{\mu\alpha} = \# \text{ of } \{0,1\} \text{ matrices with column-sums } \mu \text{ and row-sums } \alpha \quad \left(\begin{array}{l}\text{for every permuta-}\\ \text{tion } \alpha \text{ of } \rho\end{array}\right).$$

□

Corollary - $c_{\mu\rho} = c_{\rho\mu}$ $\forall \rho, \mu \vdash n$. Hence the transition matrix from the h_μ's to the m_ρ's is symmetric.

proof - The mapping

$$\varphi : \left\{ \begin{array}{l} \{0,1\}\text{-matrices with} \\ \text{row-sums } \rho \text{ and} \\ \text{column-sums } \mu \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \{0,1\}\text{-matrices with} \\ \text{row-sums } \mu \text{ and} \\ \text{column-sums } \rho \end{array} \right\}, \; A \to A^T$$

is a bijection. $\square$

We will need another technical lemma:

Lemma - $$\prod_{i,j} \left(\frac{1}{1 - x_i y_j} \right) = \sum_{\text{all partitions } \lambda} h_\lambda(x)\, m_\lambda(y).$$

proof - Recall that the generating function of $\{h_n^{(x)}\}$ is the infinite product $\prod_i \frac{1}{1 - x_i t}$, i.e.

$$\prod_i \frac{1}{1 - x_i t} = \sum_{n \geq 0} h_n(t)\, t^n.$$

Setting $t = y_j$, we obtain:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_j \left[\prod_i \frac{1}{1 - x_i \underset{\uparrow t}{y_j}} \right] = \prod_j \left[\sum_{n_j \geq 0} h_{n_j}(x)\, y_j^{n_j} \right].$$

We want to write this as a power series in y.

We notice that every monomial

$$y_{j_1}^{\boxed{n_1}}\, y_{j_2}^{\boxed{n_2}} \cdots y_{j_r}^{\boxed{n_r}}$$

appears with The same coefficient

$$h_{n_1}(x) h_{n_2}(x) \cdots h_{n_r}(x).$$

Hence we can write

$$\prod_{ij} \frac{1}{1-x_i y_j} = \sum_{n \geq 0} \sum_{\substack{(n_1, n_2, \ldots, n_r) \\ \text{partition of } n}} h_{n_1}(x) \cdots h_{n_r}(x) \, m_{(n_1 \cdots n_r)}(y) =$$

$$= \sum_{n \geq 0} \sum_{\lambda \vdash n} h_\lambda(x) \, m_\lambda(y) = \sum_{\underline{\text{all}} \text{ partitions } \lambda} h_\lambda(x) \, m_\lambda(y). \qquad \square$$