A scalar product

In this section, we define a scalar product on the space $\Lambda$ of symmetric functions.

$[\text{Here } \Lambda = \text{Span}_\mathbb{Q}\{m_\lambda : \lambda \text{ partitions}\} = \text{Span}_\mathbb{Q}\{h_\lambda : \lambda \text{ partitions}\} = \text{Span}_\mathbb{Q}\{e_\lambda : \lambda \text{ partitions}\} = \text{Span}_\mathbb{Q}\{s_\lambda : \lambda \text{ partitions}\}]$

Let $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Q}$ be the unique bilinear form that satisfies the condition

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \quad (*)$$

for every pair of partitions $\lambda$ and $\mu$.

Because $\{m_\lambda\}$, $\{h_\mu\}$ are basis of $\Lambda$, $\langle \cdot, \cdot \rangle$ is uniquely defined. The condition $(*)$ means that $\{m_\lambda\}$ and $\{h_\mu\}$ are dual basis of $\Lambda$ w.r.t. $\langle \cdot, \cdot \rangle$. It also implies that $\Lambda^n$ is orthogonal to $\Lambda^m$ for $n \neq m$. So the bilinear form $\langle \cdot, \cdot \rangle$ respects the grading $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$.

**Proposition** - The scalar product $\langle \cdot, \cdot \rangle$ is symmetric, i.e. $\langle f, g \rangle = \langle g, f \rangle \quad \forall f, g \in \Lambda$.

**Proof** - Because $\langle \cdot, \cdot \rangle$ is bilinear, it is enough to prove that $\langle f, g \rangle = \langle g, f \rangle \quad \forall f, g$ in a basis of $\Lambda$.

Choose $B = \{h_\lambda : \lambda \text{ partitions}\}$. Then we can write:

- $\langle h_\lambda, h_\mu \rangle = \langle h_\lambda, \sum_\gamma c_{\mu\gamma} m_\gamma \rangle = \sum_\gamma \langle h_\lambda, m_\gamma \rangle c_{\mu\gamma} = c_{\mu\lambda}$
and
\[ \langle h_\mu, h_\lambda \rangle = \langle h_\mu, \sum_\delta C_{\lambda \delta} m_\delta \rangle = \sum_\delta \langle h_\mu, m_\delta \rangle C_{\lambda \delta} = c_{\lambda \mu}, \]

because \( \langle h_\alpha, m_\beta \rangle = \delta_{\alpha \beta} \) for every pair of partitions \( \alpha \) and \( \beta \).

We have already observed that
\[ c_{\mu \lambda} = \text{number of \((0,1)\)-matrices with row-sums } \lambda \text{ and column-sums } \mu. \]

Hence \( \langle h_\lambda, h_\mu \rangle = \langle h_\mu, h_\lambda \rangle \) for all \( \lambda, \mu \).

\( \Rightarrow \) \( \langle , \rangle \) is symmetric. \( \square \)

\[ \begin{array}{c}
\text{We also want to show that } \langle , \rangle \text{ is positive definite.} \\
\text{Notice that it is enough to prove that there exists a basis } \{ f_\lambda \} \text{ which is self-dual with } \langle , \rangle: \\
\text{if } \langle f_\lambda, f_\mu \rangle = \delta_{\lambda \mu} \text{ for all } \lambda, \mu, \text{ then we can write} \\
\langle f, f \rangle = \sum_\lambda a_\lambda f_\lambda, \sum_\mu a_\mu f_\mu = \sum_\lambda a_\lambda^2 \geq 0 \\
\text{and } \langle f, f \rangle = 0 \iff a_\lambda = 0 \text{ for all } \lambda.
\end{array} \]

Once we prove the existence of a self-dual basis, we will have that \( \langle , \rangle \) is a well-defined scalar product on \( \Lambda \).
PLAN -

1. Understand the concept of "self-dual basis".

We will show that two basis \( \{v_\lambda\}, \{v_\lambda\}' \) of \( \Lambda \) are dual
\[\iff \langle v_\lambda, v_\beta \rangle = \delta_{\lambda \beta}, \ \forall \lambda, \beta \iff \prod_{ij} \frac{1}{1-x_i y_j} = \sum_{\lambda} v_\lambda(x) v_\lambda'(y).\]
So a self-dual basis \( \{f_\lambda\} \) is a basis of \( \Lambda \) that satisfies the condition
\[\prod_{ij} \frac{1}{1-x_i y_j} = \sum_{\lambda} f_\lambda(x) f_\lambda(y).\]

2. Give an alternative definition of Shur functions

3. Prove that the Shur functions satisfy the Cauchy formula:
\[\prod_{ij} \frac{1}{1-x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda'(y).\]
It will follow that the "new" Shur functions \( \{s_\lambda\} \) are a self-dual basis of \( \Lambda \). Hence \( \langle, \rangle \) is a scalar product.

4. Prove that the new definition of Shur functions is equivalent to the old one.
Definition Two basis \( f_{\mu 1}, f_{\nu 3} \) of \( \Lambda \) are dual if and only if \( \langle u_{\lambda}, v_{\mu} \rangle = \delta_{\lambda \mu}, \forall \lambda, \mu \).

Lemma \( f_{\nu 3} \) and \( f_{\mu 3} \) are dual if and only if

\[
\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} u_{\lambda}(x) v_{\lambda}(y).
\]

Proof: Let \( S \) be the matrix with entries

\[ S_{\nu \mu} = \langle u_{\nu}, v_{\mu} \rangle. \]

Then the basis \( f_{\lambda 1} \) and \( f_{\mu 3} \) are dual if and only if \( S = I \).

Let's try to compute the matrix \( S \).

For every partition \( \lambda \) and \( \mu \), write

\[ m_{\lambda} = \sum_{g} a_{\lambda g} u_{g}, \]
\[ h_{\mu} = \sum_{v} b_{\mu v} v_{v}, \]

and consider the matrices \( A \) and \( B \) with entries \( a_{\lambda g}, \{ b_{\mu v} \} \). Then

\[ s_{\nu \mu} = \langle m_{\lambda}, h_{\mu} \rangle = \sum_{g} a_{\lambda g} b_{\mu v} \langle u_{g}, v_{v} \rangle = \langle \text{bilinear} \rangle = \sum_{g} A_{\lambda g} \left[ \sum_{v} S_{\nu v}(B^T) v_{v} \right] = \]

\[ = \sum_{g} A_{\lambda g} \left[ \sum_{v} \langle u_{g}, v_{v} \rangle b_{\mu v} \right] = \sum_{g} A_{\lambda g} \left[ \sum_{v} S_{\nu v}(B^T) v_{v} \right] = \]
\[ = \sum_{\lambda} A_{\lambda \gamma} \left( SB^T \right)_{\gamma \mu} = (ASB^T)_{\lambda \mu}. \]

\[ \Rightarrow ASB^T = I. \]

Notice that both A and B are invertible matrices (because they are changes of basis).

\[ \Rightarrow S = A^{-1}(B^t)^{-1}. \]

So we can write:

\[ \{v_\lambda, s_\nu\} \text{ are dual} \quad \Leftrightarrow \quad S = I \quad \Leftrightarrow \quad A^{-1}(B^t)^{-1} = I \]

\[ \downarrow \]

\[ B^T A = I. \]

We will show that the condition

\[ B^T A = I \quad (\Leftrightarrow \sum_{\lambda} b_{\lambda \gamma} a_{\gamma \delta} = h_{\delta \gamma}) \]

is equivalent to the condition

\[ \prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda} u_\lambda(x) v_\lambda(y). \]

To show the equivalence, we write down the infinite product \( \prod_{i,j} \frac{1}{1-x_i y_j} \) in an alternative way.

Recall that, by a lemma proved last time,

\[ \prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda} h_\lambda(x) m_\lambda(y) = \sum_{\lambda} m_\lambda(x) h_\lambda(y). \]
Therefore we can write:

\[
\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} m_\lambda(x) h_\lambda(y) = \\
= \sum_{\lambda} \left( \sum_g \alpha_{\lambda g} u_\lambda(x) \right) \left( \sum_f b_{\lambda f} v_\lambda(y) \right) = \\
= \sum_{g,\nu} \left( \sum_\lambda b_{\lambda \nu} \alpha_{\lambda g} \right) u_\lambda(x) v_\nu(y) = \\
= \sum_{g,\nu} \left[ \sum_\lambda (B^T)_{\nu \lambda} A_{\lambda g} \right] u_\lambda(x) v_\nu(y) = \\
= \sum_{g,\nu} (B^T A)_{\nu g} u_\lambda(x) v_\nu(y).
\]

Because \( u_\lambda \) and \( v_\nu \) are basis, \( \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_f u_\lambda(x) v_\mu(y) \) if and only if \( (B^T A)_{\nu g} = \delta_{\nu g} \) \( \forall \nu, g \Leftrightarrow B^T A = I \).

We obtain:

\( \{u_\lambda\}, \{v_\nu\} \) are dual \( \Leftrightarrow \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_f u_\lambda(x) v_\mu(y) \),

as claimed. \( \square \)
STEP 2  Give an alternative definition of Schur functions.

For the moment, we work in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$, i.e., we restrict the number of variables to $n$.

For every sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ of $n$ integers $\geq 0$, let $x_\alpha$ be the monomial

$$x_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and let

$$a_\alpha(x_1, \ldots, x_n) = \det \begin{pmatrix} x_1^{\alpha_j} & x_2^{\alpha_j} & \cdots & x_n^{\alpha_j} \\ x_1^{\alpha_j} & x_2^{\alpha_j} & \cdots & x_n^{\alpha_j} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_j} & x_2^{\alpha_j} & \cdots & x_n^{\alpha_j} \end{pmatrix} =$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_1^{\alpha_{\sigma_1}} x_2^{\alpha_{\sigma_2}} \cdots x_n^{\alpha_{\sigma_n}} =$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) x^{\sigma(\alpha)}.$$

Notice that $a_\alpha$ changes sign if two of the $x_i$'s are interchanged (indeed, this corresponds to switching two columns of the matrix $(x_i^{\alpha_j})$). In particular, $a_\alpha = 0$ if two $x_i$'s are equal.

So we restrict the attention to sequences $\alpha \in \mathbb{N}^n$ which have unequal parts.
Given such a sequence \( x \), we can reorder the entries so that \( x_1 > x_2 > \ldots > x_n \). This corresponds to reordering the columns of the matrix \((x_i^T)\), so the determinant only changes by a factor \((-1)\).

\[ \Rightarrow \text{Up to sign, we can assume that } x \in \mathbb{N}^n \text{ is a sequence that satisfies } x_1 > x_2 > \ldots > x_n > 0 \text{ (i.e., a partition of } N = \sum_{i=1}^{n} a_i \text{ with strictly decreasing entries}). \]

Notice that every such sequence can be written in the form

\[ x = \lambda + \delta \]

where \( \delta = (n-1, n-2, \ldots, 1, 0) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (a_1 - (n-1), a_2 - (n-2), \ldots, a_{n-1} - 1, a_n) \). Here \( \lambda \) is again a partition (i.e., \( \lambda_1 > \lambda_2 > \ldots > \lambda_n \)) but the parts of \( \lambda \) are not necessarily distinct.

**Example:** \( n = 4 \)

\[ \lambda = (5, 3, 2, 0) \leftarrow \text{distinct parts} \]

\[ \delta = (3, 2, 1, 0) \]

\[ \lambda = (2, 1, 1, 0) \leftarrow \text{not necessarily distinct parts} \]
We are interested in the functions

\[ a_\lambda = a_{\lambda + \delta} (x_1, ..., x_n) \]

where \( \lambda \) is any partition in at most \( n \) parts

\((\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)\).

Notice that \( a_\lambda \) is a skew-symmetric function in \( x_1, ..., x_n \) of degree \( \sum_{i=1}^{n} \lambda_i \).

When \( \lambda = 0 \), we call \( a_\delta \) the **Vandermonde determinant**

\[ a_\delta = \det (x_i^j) = \det (x_i^{n-j}) = \det \begin{vmatrix}
  x_1^{n-1} & x_1^{n-2} & \cdots & x_1 \\
  x_2^{n-1} & x_2^{n-2} & \cdots & x_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  x_n^{n-1} & x_n^{n-2} & \cdots & x_n
\end{vmatrix} \]

**Claim**

\[ a_\delta (x_1, ..., x_n) = \prod_{i < j} (x_i - x_j). \]

**Proof** - Notice that \( a_\delta \) vanishes if you set \( x_i = x_j \) (because two rows of the matrix \( (x_i^{n-j}) \) become equal).

\[ \Rightarrow a_\delta (x_1, ..., x_n) \text{ is divisible by } (x_i - x_j) \forall i \neq j \]

\[ \Rightarrow a_\delta (x_1, ..., x_n) \text{ is divisible by } \prod_{i < j} (x_i - x_j). \]

Also notice that

- \( a_\delta (x_1, ..., x_n) \) is a homogeneous polynomial of degree

\[ (n-1) + (n-2) + (n-3) + \cdots + 2 + 1 + 0 = \frac{n(n-1)}{2} = \frac{n^2 - n}{2} \]

- \( \prod_{i < j} (x_i - x_j) \) is also a homogeneous polyn. of degree
\[
\frac{n^2-n}{2}
\]

\[\Rightarrow \exists c \text{ s.t. } a_{\delta}(x_1, \ldots, x_n) = c \prod_{i<j} (x_i - x_j) .
\]

Comparing the coefficients of
\[
x_1^{n-1} x_2^{n-2} \cdots x_{n-1} x_n
\]
in the two polynomials, you see that \(c = 1\).

Hence \(a_{\delta}(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)\), as claimed. \(\square\)

Remark: For all \(\lambda = \lambda + \delta\) (\(\lambda\): any partition in at most \(n\) parts), \(a_\lambda = a_{\lambda+\delta}\) is divisible by \(a_\delta\).

Indeed
\[
a_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}(x_1, \ldots, x_n) = \det(x_i^{\lambda+\delta+n-j})
\]
vanishes if \(x_i = x_j\).

\[\Rightarrow a_\lambda(x_1, \ldots, x_n) \text{ is divisible by } \prod_{i<j} (x_i - x_j) = a_\delta \text{ (in the polynomial ring } \mathbb{Q}[x_1, \ldots, x_n]\).
\]

Define
\[
s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}(x_1, \ldots, x_n)}{a_\delta(x_1, \ldots, x_n)} .
\]

Because both \(a_{\lambda+\delta}\) and \(a_\delta\) are skew-symmetric functions, the quotient \(s_\lambda\) is a symmetric function.
is an isomorphism, so the "algebraically defined" $\tilde{S}_\lambda$'s (where $\lambda$ is a partition in at most $n$ parts) are a basis of the space of symmetric functions in $x_1, \ldots, x_n$.

**Note - Defining $\tilde{S}_\lambda$** If $x_1^{d_1} \cdots x_n^{d_n}$ is a monomial in $S_n$ with coefficient $c_\lambda$, then every monomial $x_1^{j_1} \cdots x_n^{j_n}$ (with $(j_1, \ldots, j_n) \in \mathbb{N}^n$) should appear in $\tilde{S}_\lambda$ with the same coefficient. If you construct $\tilde{S}_\lambda$ according to this rule, you obtain a symmetric function with the property that $\tilde{S}_\lambda(x_1^{d_1} \cdots x_n^{d_n}) = S_\lambda(x_1, \ldots, x_n)$.

$\tilde{S}_\lambda$ is uniquely determined by $S_\lambda$. 

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