STEP 3 Show that the "algebraically defined" Shur functions satisfy the condition

$$\sum_\lambda s_\lambda(x)\, s_\lambda(y) = \prod_{i,j} \left(\frac{1}{1-x_iy_j}\right). \quad (*)$$

Cauchy formula

[Hence, the $\{s_\lambda\}$'s are a self-dual basis of Λ].

Notice that it is enough to show that

$$\sum_{\substack{\lambda \text{ partitions in} \\ \text{at most } n \text{ parts}}} s_\lambda(x_1 \dots x_n)\, s_\lambda(y_1 \dots y_n) = \prod_{\substack{i=1 \dots n \\ j=1 \dots n}} \left(\frac{1}{1-x_iy_j}\right)$$

the identity $(*)$ will follow by letting $n \to \infty$.

<u>Theorem</u> [Cauchy formula]
$$\sum_{\substack{\lambda \text{ partition} \\ \text{in at most } n \\ \text{parts}}} s_\lambda(x_1 \dots x_n)\, s_\lambda(y_1 \dots y_n) = \prod_{i,j=1 \dots n} \frac{1}{1-x_iy_j}.$$

<u>proof</u> - We show that

$$\sum_{\substack{\lambda \text{ partitions in at} \\ \text{most } n \text{ parts}}} a_{\lambda+\delta}(x_1 \dots x_n)\, a_{\lambda+\delta}(y_1 \dots y_n) = \frac{a_\delta(x_1 \dots x_n)\, a_\delta(y_1 \dots}{\prod_{i,j=1}^{n}(1-x_iy_j)}$$

by showing that both sides of the equation are equal to

$$\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1\ldots n}.$$

So the plan is to evaluate this determinant in two different ways...

CLAIM 1

$$\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1\ldots n} = \frac{a_\delta(x_1\ldots x_n)\, a_\delta(y_1\ldots y_n)}{\prod_{i,j=1\ldots n}(1-x_iy_j)}$$

for all $n \geq 1$.

Proof- By induction on n.

Base of induction: $n=1$, then $\det\left(\frac{1}{1-x_1y_1}\right) = \frac{1}{1-x_1y_1} =$

$$= \frac{\overbrace{a_\delta(x_1)}^{1}\ \overbrace{a_\delta(y_1)}^{1}}{1-x_1y_1} \quad \checkmark$$

Inductive hypothesis: True for $n-1$ variables.

Inductive step: Compute $\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1\ldots n}$.

We can write:

$$\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1\ldots n} = \det\begin{bmatrix} \boxed{\frac{1}{1-x_1y_j}}_{\,j=1\ldots n} \\ \boxed{\frac{1}{1-x_2y_j}}_{\,j=1\ldots n} \\ \vdots \\ \boxed{\frac{1}{1-x_ny_j}}_{\,j=1\ldots n} \end{bmatrix} =$$

$$= \det \begin{bmatrix} \left[\dfrac{1}{1-x_1y_j}\right]_{j=1..n} \\ \left[\dfrac{y_j\,(x_i-x_1)}{(1-x_iy_j)(1-x_1y_j)}\right] \end{bmatrix}_{\substack{i=2..n\\ j=1..n}} =$$

subtract the first row from the other ones, and use that

$$\frac{1}{1-x_iy_j} - \frac{1}{1-x_1y_j} = \frac{y_j\,(x_i-x_1)}{(1-x_iy_j)(1-x_1y_j)}$$

$$= \prod_{\substack{i=2..n\\ j=1..n}} \frac{(x_i-x_1)}{(1-x_1y_j)} \det \begin{bmatrix} 1 \; 1 \; \cdots \; 1 \\ \left[\dfrac{y_j}{1-x_iy_j}\right] \end{bmatrix}_{\substack{i=2..n\\ j=1..n}} =$$

Take out a factor $\frac{1}{1-x_1y_j}$ from every j^{th} column and a factor (x_i-x_1) from every i^{th} row, with $i=2..n$

$$= \prod_{\substack{i=2..n\\ j=1..n}} \frac{x_i-x_1}{1-x_1y_j} \det \begin{bmatrix} 1 \; 0 \; 0 \; \cdots \; 0 \\ \left[\dfrac{y_j-y_1}{(1-x_iy_j)(1-x_iy_1)}\right] \end{bmatrix}_{\substack{i=2..n\\ j=1..n}} =$$

subtract the first column and use that

$$\frac{y_j}{1-x_iy_j} - \frac{y_1}{1-x_iy_1} = \frac{y_j-y_1}{(1-x_iy_j)(1-x_iy_1)}$$

$$= \frac{1}{(1-x_1y_1)} \prod_{i=2..n} \frac{(x_i-x_1)}{(1-x_iy_1)} \prod_{j=2..n} \frac{(y_j-y_1)}{(1-x_1y_j)} \cdot \det \begin{bmatrix} 1 & 0 \; 0 \; \cdots \; 0 \\ * & \\ * & \left[\dfrac{1}{1-x_iy_j}\right] \\ * & \end{bmatrix}_{i,j=2..n}$$

Take out a factor y_j-y_1 from the j^{th} column, $\forall j \geq 2$ and a factor $\frac{1}{1-x_iy_1}$ from the i^{th} row, $\forall i \geq 2$

$$\det \begin{bmatrix} 1 & 0 \; 0 \; \cdots \; 0 \\ * & \\ * & \left[\dfrac{1}{1-x_iy_j}\right] \\ * & \end{bmatrix}_{i,j=2..n} = \det\left[\frac{1}{1-x_iy_j}\right]_{i,j=2..n}$$

now we can use induction

$$= \frac{1}{1-x_1y_1} \prod_{i=2..n} \frac{x_i - x_1}{1-x_iy_1} \prod_{j=2..n} \frac{y_j - y_1}{1-x_1y_j} \frac{a_\delta(x_2..x_n)\, a_\delta(y_2..y_n)}{\prod_{i,j=2..n} (1-x_iy_j)} =$$

$$= \frac{(-1)^n \left[\prod_{i=2}^{n} (x_1 - x_i)\right] (-1)^n \left[\prod_{j=2..n} (y_1-y_j)\right] \left[\prod_{2<i<j<n} (x_i-x_j) \prod_{2<i<j<n} (y_i-y_j)\right]}{\prod_{i,j=1..n} (1-x_iy_j)} =$$

$$= \frac{\left[\prod_{1<i<j<n} (x_i - x_j)\right]\left[\prod_{1<i<j<n} (y_i-y_j)\right]}{\prod_{i,j=1..n} (1-x_iy_j)} = \frac{a_\delta(x_1..x_n)\, a_\delta(y_1..y_n)}{\prod_{i,j=1}^{n} (1-x_iy_j)}$$

ok

CLAIM 2 $\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1..n} = \sum_{\substack{\lambda \text{ partitions in} \\ \text{at most } n \\ \text{parts}}} a_{\lambda+\delta}(x_1..x_n)\, a_{\lambda+\delta}(y_1..y_n)$

Indeed we can write:

$$\det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1..n} = \det\left(\sum_{\alpha_{ij}=0}^{\infty} (x_iy_j)^{\alpha_{ij}}\right) \rightarrow$$

When you take the determinant, you never multiply ... same row. So you can use the same index α_i for all the entries in the i^{th} row.

$$\to = \det\left(\sum_{\alpha_i=0}^{\infty} (x_i y_j)^{\alpha_i}\right)_{\substack{i=1\ldots n\\ j=1\ldots n}} =$$

$$= \det\begin{vmatrix} \boxed{\sum_{\alpha_1=1}^{\infty} x_1^{\alpha_1} y_j^{\alpha_1}}_{j=1\ldots n} \\ \boxed{\sum_{\alpha_2=1}^{\infty} x_2^{\alpha_2} y_j^{\alpha_2}}_{j=1\ldots n} \\ \vdots \\ \boxed{\sum_{\alpha_n=1}^{\infty} x_n^{\alpha_n} y_j^{\alpha_n}}_{j=1\ldots n} \end{vmatrix} =$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n \geq 0} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \det\begin{bmatrix} \boxed{y_j^{\alpha_1}} \\ \boxed{y_j^{\alpha_2}} \\ \vdots \\ \boxed{y_j^{\alpha_n}} \end{bmatrix} =$$

linearity on rows

$$= \sum_{\alpha \in \mathbb{N}^n} x^{\alpha} \det\left[y_j^{\alpha_i} \right]_{ij=1\ldots n} =$$

compositions in n parts

$$= \sum_{\alpha \in \mathbb{N}^n} x^{\alpha} \det\left[y_i^{\alpha_j} \right]_{ij=1\ldots n} = \sum_{\alpha \in \mathbb{N}^n} x^{\alpha} a_{\alpha}(y_1 \ldots y_n) =$$

the determinant is invariant under transposition

$$= \sum_{\substack{\alpha \in \mathbb{N}^n \\ \text{distinct parts}}} x^{\alpha} a_{\alpha}(y_1 \ldots y_n) =$$

$a_\alpha = 0$ if α has repeated entries

any $\alpha \in \mathbb{N}^n$ with not equal parts is a permutation of some $\lambda + \delta$, with λ = a partition in at most n parts

$$= \sum_{\substack{\lambda:\text{ partitions}\\ \text{in at most } n \text{ parts}}} \left(\sum_{\sigma \in S_n} x^{\sigma(\lambda+\delta)} \, a_{\sigma(\lambda+\delta)}(y_1 \ldots y_n) \right) =$$

$$= \sum_{\substack{\lambda:\text{ partitions in}\\ \text{at most } n \text{ parts}}} \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\lambda+\delta)} \right) a_{\lambda+\delta}(y_1 \ldots y_n) =$$

$$(\lambda+\delta) = \operatorname{sgn}(\sigma)\, a_{\lambda+\delta}$$

$$= \sum_{\substack{\lambda:\text{ partitions}\\ \text{in at most } n \text{ parts}}} a_{\lambda+\delta}(x_1 \ldots x_n)\, a_{\lambda+\delta}(y_1 \ldots y_n).$$

OK

The Cauchy's formula follows. $\square$

Corollary - The set $\{s_\lambda\}_{\lambda \text{ partition}}$ is a self-dual basis of Λ.

Corollary - the bilinear form $\langle , \rangle$ is symmetric and positive definite. So it's a scalar product on Λ.

To be postponed

Corollary - [Assume the equivalence of the two definitions of Shur functions] Then $h_\mu = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} s_\lambda$.

proof – We know that $s_\lambda = \sum_{\rho \trianglelefteq \lambda} K_{\lambda\rho}\, m_\rho$.

Take the inner product of both sides with h_μ:

$$\langle s_\lambda, h_\mu \rangle = \sum_{\rho \trianglelefteq \lambda} K_{\lambda\rho} \underbrace{\langle m_\rho, h_\mu \rangle}_{\delta_{\rho\mu}} = K_{\lambda\mu} \text{ if } \mu \trianglelefteq \lambda$$

and $= 0$ otherwise.

Then, because $\{s_\lambda\}$ is an orthonormal basis of Λ, we can write

$$h_\mu = \sum_\lambda \langle s_\lambda, h_\mu \rangle s_\lambda = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu}\, s_\lambda$$

$$\Rightarrow \boxed{h_\mu = s_\mu + \sum_{\lambda \triangleright \mu} K_{\lambda\mu}\, s_\lambda}.$$

[Because $K_{\lambda\mu} = 0$ if $\mu \triangleright \lambda$, you can also write

$h_\mu = \sum_\lambda K_{\lambda\mu}\, s_\lambda$].