Equivalence of the two definitions of Shur functions.

Recall the two definitions: Let λ be a partition in at most n parts

(I) $\displaystyle s_\lambda^{I}(x_1, x_2, \dots) = \sum_{\substack{T:\ SSYT \\ \text{of shape } \lambda}} x^T = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_\mu = \sum_\mu K_{\lambda\mu} m_\mu.$

(II) s_λ^{II} is the unique element of Λ that satisfies the condition

$$\underbrace{s_\lambda^{II}(x_1 \dots x_n 0 0 \dots)}_{} \; s_\lambda^{II}(x_1 \dots x_n) = \frac{a_{\delta+\lambda}(x_1 \dots x_n)}{a_\delta(x_1 \dots x_n)}.$$

all variables $x_{n+1}, x_{n+2}, \dots$ are set equal to zero

We know that the functions $\{s_\lambda^{II}\}$ are orthonormal wrt the inner product on Λ defined by $\langle h_{\mu_1}, m_{\mu_2} \rangle = \delta_{\mu_1 \mu_2}$.

In order to prove that $s_\lambda^{I}(x) = s_\lambda^{II}(x)$, it is enough to show that

$$s_\lambda^{II} = \sum_\mu K_{\lambda\mu} m_\mu \quad (= s_\lambda^{I}).$$

Let $s_\lambda^{II} = \sum_\mu c_{\lambda\mu} m_\mu$ be the expression of s_λ^{II} as a linear combination of the elements of the basis $\{m_\mu\}$.

Our plan is to prove that

$$c_{\lambda\mu} = K_{\lambda\mu} \quad \forall \lambda, \mu.$$

We can write:

$$\langle s_\lambda^{II}, h_{\mu'} \rangle = \langle \sum_\mu c_{\lambda\mu} m_\mu, h_{\mu'} \rangle = \sum_\mu c_{\lambda\mu} \underbrace{\langle m_\mu, h_{\mu'} \rangle}_{\delta_{\mu\mu'}} = c_{\lambda\mu'}.$$

So we really have to prove that

$$\langle s_\lambda^{II}, h_\mu \rangle = K_{\lambda\mu} \quad \forall \lambda, \mu.$$

Because the s_λ^{II}'s are an orthonormal basis, this is equivalent to showing that

$$h_\mu = \sum_\lambda \langle h_\mu, s_\lambda^{II} \rangle s_\lambda^{II} = \sum_\lambda K_{\lambda\mu} s_\lambda^{II}$$

or, by applying the involution ω, that

$$e_\mu = \sum_\lambda K_{\lambda\mu} \, \omega(s_\lambda^{II}).$$

We get:

CLAIM: $s_\lambda^{I} = s_\lambda^{II} \iff e_\mu = \sum_\lambda K_{\lambda\mu} \, \omega(s_\lambda^{II})$.

We will prove that

1. $\omega(s_\lambda^{II}) = s_{\lambda'}^{II}$, with λ' the dual partition

2. $e_\mu = \sum_\lambda K_{\lambda\mu} s_{\lambda'}^{II} = \sum_\lambda K_{\lambda'\mu} s_\lambda^{II}$.

The equivalence between s_λ^{I} and s_λ^{II} will follow from the claim.

$\boxed{\text{EQUIVALENCE OF THE TWO DEFINITIONS OF SHUR FUNCTIONS}}$ PART 1 $\longrightarrow$

Proving That $\omega(s_\lambda^{II}) = s_{\lambda'}^{II}$ ... (The main idea)

- We do a proof in several steps:

(a) Show That

$$s_\lambda^{II}(x_1 \dots x_n) = \det\left(h_{\lambda_i - i + j}\right)_{ij=1\dots n} \quad (*)$$

for every partition of n in at most n parts.

[Here $s_\lambda^{II}(x_1 \dots x_n) = a_{\delta+\lambda}/a_\delta$].

(b) the identity (*) is called The Jacobi-Trudi identity. It shows That $s_\lambda^{II}(x_1 \dots x_n)$ is a minor of the matrix

- $$H = \left(h_{\ell-k}\right)_{\ell,k\geq 0}$$

Because H is the inverse of The matrix $E = \left((-1)^{k-\ell} e_{k-\ell}\right)_{k,\ell\geq 0}$, every minor of H can be expressed as a minor of E^T. It follows that

$$s_\lambda^{II}(x_1 \dots x_n) = \det\left(e_{\lambda'_i - i + j}\right)_{ij=1\dots n}$$

with λ' The dual partition.

(c) Because The involution ω switches h_a with e_a $\forall a \geq 0$, and respects multiplication, we can write

- $$\omega\left(s_\lambda^{II}(x_1 \dots x_n)\right) = \omega\left(\det\left(h_{\lambda_i - i + j}\right)\right) = \det\left(e_{\lambda_i - i + j}\right) = s_{\lambda'}^{II}(x_1 \dots x_n).$$

Hence $\omega(s_\lambda^{II}) = s_{\lambda'}^{II}$, as claimed. Details follow...

Proving that $\omega(s_\lambda^{II}) = (s_{\lambda'}^{u})$ → The details !!!

The Jacobi-Trudi Identity

Theorem – If $\tilde{\lambda}$ is a partition in at most n parts, then

$$s_{\tilde{\lambda}} = \det\left(h_{\tilde{\lambda}_i - i + j}\right)_{i,j=1\ldots n}$$

with the convention that $h_r = 0$ if $r<0$ (and $h_0 = 1$).

Proof | Set $X = (x_1 \ldots x_n)$, $y = (y_1 \ldots y_n)$ | To prove this identity, we multiply both sides of the equation

$$\sum_{\substack{\lambda \\ \text{in at most } n \text{ parts}}} h_\lambda(x)\, m_\lambda(y) = \sum_{\substack{\lambda \\ \text{in at most} \\ n \text{ parts}}} s_\lambda(x)\, s_\lambda(y) \quad \left(= \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} \right)$$

by $a_\delta(y)$, and we compare the coefficients of $y^{\delta+\tilde{\lambda}}$ in $\left[\sum_{\substack{\lambda \\ \#\text{parts} \le n}} h_\lambda(x) m_\lambda(y)\right] a_\delta(y)$ and $\left[\sum_{\substack{\lambda \\ \#\text{parts} \le n}} s_\lambda(x) s_\lambda(y)\right] a_\delta(y)$.

Recall that $\delta = (n-1, n-2, \ldots, 1, 0)$ and

- $a_\delta(y) = \det\left(x_i^{\delta_j}\right) = \det\left(x_i^{n-j}\right) = \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma)\, y^{\sigma(\delta)}$

- $$\sum_{\substack{\lambda \\ \#\text{parts} \le n}} h_\lambda(x)\, m_\lambda(y) = \sum_{\substack{\lambda \\ \#\text{parts} \le n}} h_\lambda(x) \left[\sum_{\substack{\alpha:\text{ distinct} \\ \text{permutation} \\ \text{of } \lambda}} y^\alpha \right] =$$

$$= \sum_{\lambda\ \#\text{parts} \le n} \ \sum_{\substack{\alpha:\text{ distinct} \\ \text{permut. of } \lambda}} h_\alpha(x)\, y^\alpha =$$

if α is a permutation of λ then $h_\lambda(x) = h_\alpha(x)$

$$= \sum_{\beta \in \mathbb{N}^n} h_\beta(x)\, y^\beta .$$

(all compositions in at most n parts)

So we get:

$$a_\delta(y)\left[\sum_\lambda h_\lambda(x)\, m_\lambda(y)\right] = \left[\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, y^{\sigma(\delta)}\right] \cdot \left[\sum_{\beta\in \mathbb{N}^n} h_\beta(x)\, y^\beta\right] = \sum_{\substack{\sigma\in S_n\\ \beta\in\mathbb{N}^n}} h_\beta(x) \operatorname{sgn}(\sigma)\, y^{\beta+\sigma(\delta)} =$$

$$= \sum_{\sigma\in S_n}\left[\operatorname{sgn}\sigma \sum_{\beta\in\mathbb{N}^n} h_\beta(x)\, y^{\beta+\sigma(\delta)}\right].$$

Let's find the coefficient of $y^{\tilde\lambda+\delta}$ in this expression.

If $\beta+\sigma(\delta)=\tilde\lambda+\delta$ then $\beta=\tilde\lambda+\delta-\sigma(\delta)$. The desired coefficient is therefore

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, h_{\tilde\lambda+\delta-\sigma(\delta)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n h_{\tilde\lambda_i+n-i-(n-\sigma(i))} =$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n h_{\tilde\lambda_i - i+\sigma(i)}$$

$$= \det\left(h_{\tilde\lambda_i-i+j}\right).$$

Next, we compute $a_\delta(y)\left[\sum_{\lambda:\#\text{parts}\le n} s_\lambda(x)\, s_\lambda(y)\right]$ and we extract the coefficient of $y^{\lambda+\delta}$ from this expression.

$$a_\delta(y)\left[\sum_{\substack{\lambda\\ \#\text{parts}\le n}} s_\lambda(x)\, s_\lambda(y)\right] = a_\delta(y)\left[\sum_{\substack{\lambda\\ \#\text{parts}\le n}} s_\lambda(x)\,\frac{a_{\delta+\lambda}(y)}{a_\delta(y)}\right] =$$

$$= \sum_{\substack{\lambda\\ \#\text{parts}\le n}} s_\lambda(x)\, a_{\delta+\lambda}(y).$$

Because $y^{\tilde\lambda+\delta}$ appears in $a_{\delta+\tilde\lambda}(y)$ with coefficient 1, the coefficient of $y^{\tilde{\lambda}+\delta}$ in $a_\delta(y) \sum_{\lambda : \# \text{parts} \le n} s_\lambda(x)\, s_\lambda(y)$ is equal to $s_{\tilde{\lambda}}(x)$.

therefore we get:

$$s_{\tilde{\lambda}}(x_1 \dots x_n) = \det \left[h_{\tilde{\lambda}_i - i + j}(x_1 \dots x_n) \right]$$

as claimed. $\square$

For every partition in at most n parts

__Corollary 1__ $s_\lambda = \det \left(e_{\mu_j + i - j} \right)_{i,j = 1 \dots n}$

where μ is the dual partition.

__Sketch of the proof__ - Consider the (infinite) matrices

$$H = \left(h_{i-j} \right)_{i,j \ge 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ h_1 & 1 & 0 & 0 & \cdots \\ h_2 & h_1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$E = \left((-1)^{i-j} e_{i-j} \right)_{i,j \ge 0}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -e_1 & 1 & 0 & 0 & \cdots \\ e_2 & -e_1 & 1 & 0 & \cdots \\ -e_3 & e_2 & -e_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Because $\sum_{k=0}^{\ell} h_a e_{k-a} (-1)^{k-a} = 0 \quad \forall \ell \ge 1$, $HE = I$.

We know that $s_\lambda = \det \left[h_{\lambda_i - i + j} \right]_{i,j = 1 \dots n}$ is a minor of H.

More precisely, it is the minor corresponding to the row indices

$$\lambda_n + 1,\ \lambda_{n-1} + 2,\ \dots,\ \lambda_1 + n$$

and the column indices

$$1, 2, \dots, n.$$

Because $E = H^{-1}$, this minor of H is equal to the complementary cofactor of E^T.

<u>Lemma</u> If $\lambda = (\lambda_1 \dots \lambda_n)$ and $\mu = (\mu_1 - \mu_e)$ are conjugate partitions, then the sets

$$\{\lambda_i + n + 1 - i \ ; \ 1 \le i \le n\}$$

and

$$\{n + j - \mu_j \ : \ 1 \le j \le e\}$$

form a disjoint union of the sets $\{1 \dots n+e\}$.

<u>Proof</u>- Omitted. See I.G. MacDonald, "Symmetric Functions and Hall polynomials", Clarendon press, Oxford, 1979.

<u>Lemma</u> If $A = (a_{ij})$ is an $r \times r$ matrix and $S = (s_1 \dots s_k)$, $T = (t_1 \dots t_k)$ are two sequences of k distinct integers from $\{1 \dots r\}$, let A_{TS} be the corresponding minor (ie. the determinant of the $k \times k$ matrix with $(i,j)^{th}$ entry $a_{s_i t_j}$ -

Let $S' = (s'_1 \ldots s'_{r-k})$, $T' = (t'_1 \ldots t'_{r-k})$ be complementary sets, and let $B = A^{-1}$.

Then

$$A_{S,T} = \varepsilon (\det A)\, B_{T'S'} = \varepsilon \det A \,(B^T)_{S'T'}$$

Here ε is the sign of the permutation of $\{1 \ldots r\}$ that takes $\{S, S'\}$ into $\{T, T'\}$.

<u>proof</u> - Omitted. See Fulton and Harris, "Representation Theory", p. 464.

Apply these results to the matrix

$$A = (H_{ij})_{ij=1\ldots n+e} \; ; \quad B = A^{-1} = (E_{ij})_{ij=1\ldots n+e}$$

$$S = \{\lambda_i + n + 1 - i \; : \; 1 \le i \le n\}$$

$$T = \{n, n-1, \ldots, 2, 1\} = \{n+1-j \; : \; j = 1 \ldots n\}$$

$$S' = \{n + j - \mu_j \; : 1 \le j \le e\}$$

$$T' = \{n+1, n+2, \ldots, n+e\} = \{n+j \; : \; j = 1 \ldots e\}$$

then

$$A_{S,T} = \det\left(h_{(\lambda_i + n + 1 - i) - (n+1-j)}\right) = \det\left(h_{\lambda_i + j - i}\right) = s_\lambda$$

and

$$(B^T)_{S',T'} = \det\left((-1)^{n+i-(n+j-\mu_j)} e_{n+i-(n+j-\mu_j)} \right) = \det\left((-1)^{\mu_j + i - j} e_{\mu_j + i - j} \right).$$

$$= \det\left(\underset{\substack{\uparrow \\ \text{common To} \\ \text{The } j^{\text{Th}} \text{ column}}}{(-1)^{\mu_j - j}} \quad \underset{\substack{\downarrow \\ \text{common To The } i^{\text{Th}} \text{ row}}}{(-1)^{i}} \quad e_{\mu_j + i - j} \right) =$$

$$= \left[\prod_{i=1}^{n} (-1)^{i} \right] \left[\prod_{j=1}^{\ell} (-1)^{\mu_j - j} \right] \det\left(e_{\mu_j + i - j} \right)$$

$$= (-1)^{n} (-1)^{n} (-1)^{\sum_{j=1}^{\ell} \mu_j} \det\left(e_{\mu_j + i - j} \right) =$$

$$= (-1)^{d} \det\left(e_{\mu_j + i - j} \right).$$

μ is dual To λ

so $\sum_{j=1}^{\ell} \mu_j = \sum_{k=1}^{n} \lambda_k$.

Call iT d.

We also need To look at ε = sign of the permutation of $\{1, \ldots, n+\ell\}$ that Takes

$$\{ \underbrace{\lambda_1 + n, \lambda_2 + n - 1, \ldots, \lambda_n + 1}_{S}, \underbrace{n+1-\mu_1, n+2-\mu_2, \ldots, n+\ell-\mu_\ell}_{S'} \}$$

into

$$\{ \underbrace{n, n-1, n-2, \ldots, 1}_{T}, \underbrace{n+1, n+2, \ldots, n+\ell}_{T'} \}.$$

IT can be showed that This sign is equal To $(-1)^{d}$.

Hence

$$s_\lambda = A_{TS} = (-1)^{d} (\det A) \left((-1)^{d} \det\left(e_{\mu_j + i - j} \right) \right) =$$
$$= (\det A) \left(\det\left(e_{\mu_j + i - j} \right) \right) = \det\left(e_{\mu_j + i - j} \right)$$

because

$$\det A = \det(h_{i-j})_{ij=1\ldots n+e} = 1.$$

[Recall that $h_k = 0$ for $k<0$ and $h_0 = 1$, so A is triangular with one's on the diagonal].

$$\Rightarrow \boxed{s_\lambda = \det(e_{\mu_j+i-j})_{ij=1\ldots n} \text{ with } \mu = \text{dual partition.}}$$

<u>Corollary</u> $\omega(s_\lambda) = s_\mu$, where μ is the dual partition.

<u>proof</u> - $\omega(s_\lambda) = \omega\left(\det(h_{\lambda_i - i+j})_{ij=1\ldots n}\right) =$

$$= \det(e_{\lambda_i-i+j})_{ij=1\ldots n} =$$

ω carries h_k into e_k for all $k \geq 0$

det is invariant under transposition

$$= \det(e_{\lambda_j-j+i})_{ij=1\ldots n} =$$

$= s_\mu$, with μ the dual partition. $\square$

Equivalence of the Two definitions of Shur functions (PART ②).

theorem – Let μ be any partition. (μ has an arbitrary # of parts).

Then
$$e_\mu(x_1, x_2, \dots, x_n) = \sum_{\substack{\lambda:\ \text{at most} \\ n \text{ parts}}} K_{\lambda'\mu} \frac{a_{\lambda+\delta}(x_1 \dots x_n)}{a_\delta(x_1 \dots x_n)}.$$

[We have denoted by λ' The dual partition of λ]_

proof – We will show That
$$e_\mu a_\delta = \sum_\lambda K_{\lambda'\mu} a_{\lambda+\delta}.$$

Since both sides are skew-symmetric, it is enough To show That the coefficient of $x^{\lambda+\delta}$ in $e_\mu a_\delta$ is equal To $K_{\lambda'\mu}$, for all λ. Here $x = \{x_1, x_2, \dots, x_n\}$. How do we get $x^{\lambda+\delta}$? Let's look at $e_\mu a_\delta$.

Notice that
$$\underset{\mu = \mu_1, \mu_2, \dots}{e_\mu a_\delta} \underset{\pi}{=} (((a_\delta e_{\mu_1}) e_{\mu_2}) e_{\mu_3} \dots).$$

Every partial product $a_\delta e_{\mu_1} \dots e_{\mu_k}$ is skew-symmetric, so it only involves monomials $x_1^{i_1} \dots x_n^{i_n}$ with all exponents i_j distinct.

In other words, if β is any composition with some equal parts, Then The coeff. of x^β in $a_\delta e_{\mu_1} \dots e_{\mu_k}$

To get the term $x^{\delta+\lambda}$ in

$$a_\delta e_\mu = a_\delta e_{\mu_1} e_{\mu_2} e_{\mu_3} \cdots$$

we start from the term x^δ in a^δ, and we successively multiply it by a monomial x^{α_i} in e_{μ_i} $(i = 1, 2, 3, \dots)$, keeping the exponents distinct (← strictly decreasing), until we obtain $x^{\delta+\lambda}$. The coefficients of $x^{\delta+\lambda}$ in $a_\delta e_\mu$ is the number of ways to do this.

Suppose that $(\alpha_1, \alpha_2, \dots)$ is a sequence of compositions s.t.

$$x^{\lambda+\delta} = x^\delta x^{\alpha_1} x^{\alpha_2} \cdots$$

and x^{α_i} is a monomial in e_{μ_i} with the property described above (⇔ the exponents of $x^\delta x^{\alpha_1} \cdots x^{\alpha_i}$ are strictly decreasing).

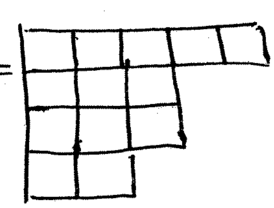
To each such sequence, we associate a semistandard Young Tableau of shape λ' and content μ, as follows:

if x^{α_i} contains the variable x_j (necessarily with exponent 1 because x^{α_i} is a monomial in e_{μ_i}), then set an i in the j^{th} column of T.

<u>Example</u> $n = 4$, $\lambda = (5\,3\,3\,2)$, $\lambda' = (4\,4\,3\,1\,1)$

$\mu = (3\,2\,2\,2\,2\,1\,1)$.

$\lambda =$



$\delta = (4-1, 4-2, 4-3, 4-4) = (3,2,1,0)$

$\Rightarrow \lambda + \delta = (5,3,3,2) + (3,2,1,0) = (8,5,4,2).$

We start from the monomial

$$x^\delta = x_1^3 x_2^2 x_3 .$$

We need to multiply x^δ by a monomial x^{α_1} in $e_{\mu_1} = e_3 = \sum_{i<j<k} x_i x_j x_k$ so that the exponents of $x^\delta x^{\alpha_1}$ are strictly increasing.

Of course we have many choices for x^{α_1}.

We choose $\alpha_1 = (1,1,1)$, i.e. $x^{\alpha_1} = x_1 x_2 x_3$.

$\Rightarrow x^\delta x^{\alpha_1} = x_1^4 x_2^3 x_3^2 .$

Now we need to multiply $x^\delta x^{\alpha_1}$ by a monomial x^{α_2} from $e_{\mu_2} = e_2 \left(= \sum_{i<j} x_i x_j \right)$ so that the exponent of $x^\delta x^{\alpha_1} x^{\alpha_2}$ are decreasing.

Say that we choose $\alpha_2 = (1,1)$, ie $x^{\alpha_2} = x_1 x_2$.

$\Rightarrow x^\delta x^{\alpha_1} x^{\alpha_2} = x_1^5 x_2^4 x_3^2 .$

Proceding in a similar way, we can choose

- $\alpha_3 = (0,0,1,1) \Rightarrow x^{\alpha_3} = x_3 x_4 \Rightarrow x^\delta x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} = x_1^5 x_2^4 x_3^3 x_4$
- $\alpha_4 = (1,1) \Rightarrow x^{\alpha_4} = x_1 x_2 \Rightarrow x^\delta x^{\alpha_1} \cdots x^{\alpha_4} = x_1^6 x_2^5 x_3^3 x_4$
- $\alpha_5 = (1,0,0,1) \Rightarrow x^{\alpha_5} = x_1 x_4 \Rightarrow x^\delta x^{\alpha_1} \cdots x^{\alpha_5} = x_1^7 x_2^5 x_3^3 x_4^2$
- $\alpha_6 = (1) \Rightarrow x^{\alpha_6} = x_1 \Rightarrow x^\delta x^{\alpha_1} \cdots x^{\alpha_6} = x_1^8 x_2^5 x_3^3 x_4^2$

• $\alpha_7 = (0,0,1) \Rightarrow x^{\alpha_7} = x_3 \Rightarrow x^0 x^{\alpha_1} \cdots x^{\alpha_7} = x_1^8 x_2^5 x_3^4 x_2^2.$

• Now let's construct the SSYT associated to the sequence

$$
\begin{array}{ll}
x^{\delta} = x_1^3 x_2^2 x_3 & \\
\downarrow & \alpha_1 = (1,1,1) \\
x_1^4 x_2^3 x_3^2 & \\
\downarrow & \alpha_2 = (1,1) \\
x_1^5 x_2^4 x_3^2 & \\
\downarrow & \alpha_3 = (0,0,1,1) \\
x_1^5 x_2^4 x_3^3 x_4 & \\
\downarrow & \alpha_4 = (1,1) \\
x_1^6 x_2^5 x_3^3 x_4 & \\
\downarrow & \alpha_5 = (1,0,0,1) \\
x_1^7 x_2^5 x_3^3 x_4^2 & \\
\downarrow & \alpha_6 = (1) \\
x_1^8 x_2^5 x_3^3 x_4^2 & \\
\downarrow & \alpha_7 = (0,0,1). \\
x_1^8 x_2^5 x_3^4 x_4^2 = & \\
= x^{\delta+\lambda} &
\end{array}
$$

remember the rule: if x^{α_i} contains x_j (ie α_i has a 1 in the j^{th} position) then set an i in the j^{th} column of T

the j^{th} column of T contain all the i's with the property that the j^{th} entry of α_i is $= 1$

• the first column contains i if x^{α_i} contains x_1 (i.e. if α_i has a 1 in position one).

$\Rightarrow$ the first column contains 1, 2, 4, 5, 6

$\Rightarrow$ the first column is $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline\end{array}$.

the second column contains i if x^{α_i} contains x_2 (i.e. if α_i has a 1 in position two).

• $\Rightarrow$ the second column contains 1, 2, 4.

$\Rightarrow$ the second column is $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array}$.

Similarly we find that

- The third column is $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 7 \\ \hline \end{array}$

- The fourth column is $\begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline \end{array}$

and there are no other columns.

$\Rightarrow$ the SSYT associated to the sequence $(\alpha_1 \alpha_2 \dots \alpha_7)$ is

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 & 5 \\ \hline 4 & 4 & 7 \\ \cline{1-3} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$$

Notice that T has shape $(4,4,3,1,1) = \lambda'$ and content $(3,2,2,2,2,1,1) = \mu$.

The map $(\alpha_1, \alpha_2, \dots) \to T = T(\alpha_1, \alpha_2, \dots)$ gives a bijection between the ways of building up $x^{\lambda+\delta}$ in $a_\delta e_\mu$ and the set of SSYT of shape λ' (=dual of λ) and content μ.

$\Rightarrow$ the coefficient of $x^{\lambda+\delta}$ in $a_\delta e_\mu$ is equal to the Kostka number $K_{\lambda'\mu}$. This completes the proof.

$\square$

Corollary -

1) The two definitions of Shur functions agree. Hence

$$s_\lambda(x_1 \dots x_n) = \frac{a_{\lambda+\delta}}{a_\lambda}(x_1 \dots x_n) = \sum_{\substack{\mu \text{ at most} \\ n \text{ parts}}} K_{\lambda\mu}\, m_\mu(x_1 \dots x_n)$$

2) The shur functions

$$s_\lambda(x_1, x_2, \dots) = \sum_{\substack{T:\ \text{SSYT} \\ \text{of shape } \lambda}} x^T$$

are an orthonormal basis of Λ.