# Another basis of $\Lambda$: Power sums

- For all $r \geq 1$, define the $r^{\text{th}}$ power sum

$$p_r(x) = m_{(r)} = \sum_{i \geq 1} x_i^r = x_1^r + x_2^r + x_3^r + \cdots.$$

The generating function for the $p_r$'s is

$$P(t) = \sum_{r \geq 1} p_r(x)\, t^{r-1} = \sum_i \frac{x_i}{1 - x_i t}.$$

This can be proved easily:

$$\sum_i \frac{x_i}{1-x_i t} = \sum_i x_i \left[\frac{1}{1-x_i t}\right] = \sum_i x_i \left[\sum_{n_i \geq 0} (x_i t)^{n_i}\right] =$$

$$= \sum_i \sum_{n_i \geq 0} x_i^{n_i+1}\, t^{n_i}.$$

- For every fixed $n \geq 0$, the coefficient of $t^n$ in this expression is given by

$$\sum_i x_i^{n+1} = p_{n+1}(x).$$

So we can write

$$\sum_i \frac{x_i}{1-x_i t} = \sum_{n \geq 0} p_{n+1}(x) t^n = \sum_{r \geq 1} p_r(x)\, t^{r-1}, \text{ as claimed.} \quad \square$$

<u>Lemma</u> ① $P(t) = \dfrac{H'(t)}{H(t)}$

② $P(-t) = \dfrac{E'(t)}{E(t)}$.

- Proof ① By a direct computation. Recall that $H(t) = \prod_{i \geq 1} \left(\frac{1}{1-x_i t}\right)$

$$\Rightarrow \frac{d}{dt}\left(\ln H(t)\right) = \frac{H'(t)}{H(t)} = \frac{1}{\prod_{i \geq 1}(1-x_i t)}\, \frac{d}{dt}\left[\prod_{i \geq 1} \frac{1}{1-x_i t}\right] =$$

$$= \frac{1}{\prod_{i\geq 1}(1-x_it)} \sum_{i=1}^{\infty} \left(\prod_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{1-x_jt}\right)\left(+\frac{x_i}{(1-x_it)^2}\right) =$$

$$= \frac{1}{\left[\prod_{i\geq 1}(1-x_it)\right]} \sum_{i=1}^{\infty} \boxed{\left(\prod_{j=1}^{\infty} \frac{1}{1-x_jt}\right)} \left(\frac{x_i}{1-x_it}\right)$$

$$= \frac{\cancel{\left[\prod_{j=1}^{\infty} \frac{1}{1-x_jt}\right]}}{\cancel{\left[\prod_{i=1}^{\infty} \frac{1}{1-x_it}\right]}} \sum_{i=1}^{\infty} \frac{x_i}{1-x_it} = \sum_{i=1}^{\infty} \frac{x_i}{1-x_it} = P(t). \checkmark$$

(2) Recall That $E(t) = \prod_{i\geq 1}(1+x_it)$.

$$\Rightarrow \frac{E(+t)'}{E(t)} = \frac{1}{\prod_{i\geq 1}(1+x_it)} \frac{d}{dt}\left[\prod_{i\geq 1}(1+x_it)\right] =$$

$$= \frac{1}{\prod_{i\geq 1}(1+x_it)} \sum_{i\geq 1} \left(\prod_{\substack{j\geq 1\\ j\neq i}} (1+x_it)\right) \frac{d}{dt}(1+x_it) =$$

$$= \frac{1}{(1+x_it)} \cdot x_i = \left(\frac{x_i}{1-x_it}\right)\Big|_{-t} = P(-t). \checkmark$$

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Corollary 1. For all $r \geq 1$, $\omega(p_r) = (-1)^{r-1} p_r$

proof - We can write

w switches H and E

$$\sum_{r\geq 1} \omega(p_r)t^{r-1} = \omega(P(t)) = \omega\left(\frac{H(t)}{H'(t)}\right) = \frac{E(t)}{E'(t)} =$$

$$= P(-t) = \sum_{r\geq 1} p_r(-t)^{r-1} = \sum_{r\geq 1} p_r(-1)^r t^{r-1}$$

$$\Rightarrow \omega(p_r) = (-1)^r p_r.$$

**Corollary 2** For every $n$, $h_n = \sum_{\lambda \vdash n} \frac{p^\lambda}{z^\lambda}$.

If $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n}) \vdash n$ is a partition of $n$ with $m_j$ parts of length $j$, then

$$z_\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n} \, m_1! \, m_2! \dots m_n!.$$

[Notice that $z_\lambda$ is the size of the conjugacy class of $S_n$ associated to $\lambda$].

**Proof** - Because $P(t) = \frac{H'(t)}{H(t)}$ " $\frac{d}{dt}(\ln H(t))$ ", we can write:

$$\ln[H(t)] = \int P(t)\,dt \Rightarrow H(t) = \exp\left(\int P(t)\,dt\right).$$

Write $P(t) = \sum_{k \geq 0} p_{k+1}(x) t^k$.

Then

$$\int P(t)\,dt = \sum_{k \geq 0} p_{k+1}(x) \frac{t^{k+1}}{k+1} = \sum_{j \geq 1} p_j(x) \frac{t^j}{j}.$$

therefore:

$$H(t) = \exp\left(\int P(t)\,dt\right) = \exp\left(\sum_{j \geq 1} p_j(x) \frac{t^j}{j}\right) = \prod_{j \geq 1} \exp\left(p_j(x) \frac{t^j}{j}\right) =$$

$$= \prod_{j \geq 1} \left( \sum_{m_j = 0}^{\infty} \frac{[p_j(x) t^j / j]^{m_j}}{m_j!} \right) = \prod_{j \geq 1} \left( \sum_{m_j \geq 0} \frac{p_j^{m_j} \, t^{j m_j}}{j^{m_j} \, m_j!} \right)$$

We notice that

$$p_\lambda = p_{(1^{m_1} 2^{m_2} \ldots n^{m_n})} = \prod_{j=1}^{n} (p_j)^{m_j}$$

assume $m_j = 0 \ \forall j > n$

appears in this expression with coefficient

$$\frac{\prod_{j\geq 1} t^{j m_j}}{\prod_{j \geq 1} j^{m_j} m_j!} = \frac{t^{\overbrace{\sum_{j\geq 1} j m_j}^{= n \text{ because } \lambda \vdash n}}}{\underbrace{\prod_{j\geq 1} j^{m_j} m_j!}_{= z_\lambda \text{ (by definition of } z_\lambda)}} = \frac{t^n}{z_\lambda}.$$

So we can write

$$H(t) = \sum_{n\geq 0} \sum_{\lambda \vdash n} \frac{p_\lambda(x) t^n}{z_\lambda} = \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} \frac{p_\lambda(x)}{z_\lambda} \right) t^n.$$

Because $H(t)$ is the generating function for $\{h_n(x)\}_{n\geq 0}$, we also get:

$$h_n(x) = \sum_{\lambda \vdash n} \frac{p_\lambda(x)}{z_\lambda}. \quad \blacksquare$$

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As a consequence, we obtain that $\forall \mu \vdash m$, we can always write $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_r}$ as a linear combination of symmetric power sums corresponding to partitions of $m$, with rational coefficients

$\Rightarrow$ The set $\{p_\lambda\}_{\lambda \vdash m}$ spans $\Lambda^m_{\mathbb{Q}}$ $\Rightarrow$ the set $\{p_\lambda\}$ spans $\Lambda_{\mathbb{Q}}$.

emark- We notice that the set of power sums are a basis for the ring of symmetric functions over $\mathbb{Q}$ (and $\mathbb{R}, \mathbb{C} \ldots$) but not over $\mathbb{Z}$.

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**Theorem** - The set $\left\{ \frac{p_\lambda}{\sqrt{z_\lambda}} \right\}_\lambda$ is an orthonormal basis of $\Lambda$.

**proof** - the statement of the theorem is equivalent to the condition

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_\lambda \frac{p_\lambda(x)}{\sqrt{z_\lambda}} \frac{p_\lambda(y)}{\sqrt{z_\lambda}} .$$

Denote by $xy$ the set of variables $x_i y_j$.
Then we can write:

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \left[ \sum_{n \geq 0} h_n(xy) t^n \right]_{t=1} = \sum_{n \geq 0} h_n(xy) =$$

Recall that
$\prod_k \frac{1}{1 - w_k t} = \sum h_n(w) t^n$
for every variable $w = \{w_k\}$

$$= \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} \frac{p_\lambda(xy)}{z_\lambda} \right) =$$

$$= \sum_{\text{all } \lambda} \frac{p_\lambda(xy)}{z_\lambda} =$$

for all $k$
$p_k(xy) = \sum_{i,j} (x_i y_j)$
$= \left( \sum_i x_i^k \right) \left( \sum_j y_j^k \right) =$
$= p_k(x) p_k($

$$= \sum_{\text{all } \lambda} \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda} = \sum_{\text{all } \lambda} \frac{p_\lambda(x)}{\sqrt{z_\lambda}} \frac{p_\lambda(y)}{\sqrt{z_\lambda}} . \quad \blacksquare$$

<u>Remark</u> - Saying that the set $\left\{ \frac{p_\lambda}{\sqrt{z_\lambda}} \right\}$ is orthonormal is equivalent to the condition

$$\langle p_\lambda , p_\mu \rangle = z_\lambda \, \delta_{\lambda\mu} \quad \forall \lambda, \mu .$$

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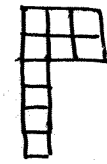
The symmetric power functions $\{p_\lambda\}$ are not an orthonormal basis of $\Lambda$, but they play a huge role in understanding the relation between $\Lambda$ and the representation theory of the symmetric group.

# The connection between symmetric functions and irreducible representations of the symmetric group

For every partition $\pi$ of $n$, write

$$\pi = (1^{\ell_1}, 2^{\ell_2}, \dots, n^{\ell_n})$$

with $\ell_j = \#$ of parts of $\pi$ of length $j$. [Example: $1^4 2^0 3^2 \leftrightarrow$  ].

$\pi$ represents a conjugacy class of $S_n$ of size

$$\frac{n!}{1^{\ell_1} \cdot 2^{\ell_2} \cdots n^{\ell_n} \, \ell_1! \, \ell_2! \cdots \ell_n!} = \frac{n!}{z_\pi}.$$

This is the constant introduced before ... (see power functions)

[In more standard notations, $\pi = (\underbrace{n, \dots, n}_{\ell_n}, \underbrace{n-1, \dots, n-1}_{\ell_{n-1}} \dots, \underbrace{2 \dots 2}_{\ell_2}, \underbrace{1 \dots 1}_{\ell_1})$].

Consider the power symmetric function associated to $\pi$ =

$$P_\pi(x_1, x_2, \dots, x_n) = (x_1^n + \dots + x_n^n)^{\ell_n} (x_1^{n-1} + x_2^{n-1} + \dots + x_n^{n-1})^{\ell_{n-1}} \cdots (x_1 + x_2 + \dots + x_n)^{\ell_1} =$$

$$= \prod_{j=1}^{n} (x_1^j + x_2^j + \dots + x_n^j)^{\ell_j}.$$

$P_\pi$ is a symmetric function of degree

$$\deg(P_\pi) = \sum_{j=1}^{n} j \ell_j = n$$

in the variables $x_1, x_2, \dots, x_n$.

So we can express $P_\pi$ as a linear combination of any basis of $\Lambda^n[x_1 \ldots x_n]$.

In particular, we can write:

$$P_\pi(x_1 \ldots x_n) = \sum_{\lambda \vdash n} \psi_\lambda(\pi)\, m_\lambda(x_1 \ldots x_n)$$

and

$$P_\pi(x_1 \ldots x_n) = \sum_{\lambda \vdash n} \omega_\lambda(\pi)\, s_\lambda(x_1 \ldots x_n)$$

for some coefficients $\{\psi_\lambda(\pi)\}$ and $\{\omega_\lambda(\pi)\}$ in $\mathbb{Q}$.

We will prove that

$$\boxed{\psi_\lambda(\pi)} = \chi_{M^\lambda}(\pi) =$$ the character of the permutation module $M^\lambda$ evaluated at the conjugacy class $\pi$

$$\boxed{\omega_\lambda(\pi)} = \chi_{S^\lambda}(\pi) =$$ The character of the Specht module $S^\lambda$ evaluated at the conjugacy class $\pi$.

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<u>Remark</u> – Because

$$P_\pi(x_1 \ldots x_n) = \sum_{\lambda \vdash n} \psi_\lambda(\pi)\, m_\lambda = \sum_{\lambda \vdash n} \psi_\lambda(\pi) \Big( \sum_{\substack{\alpha \text{ distinct} \\ \text{permutations} \\ \text{of } \lambda}} x^\alpha \Big)$$

we can characterize $\psi_\lambda(\pi)$ as the coefficient of $x^\lambda$ inside $P_\pi(x_1 \ldots x_n)$.

Similarly, writing $s_\lambda(x_1 \ldots x_n) = \dfrac{a_{\lambda+\delta}(x_1 \ldots x_n)}{a_\delta(x_1 \ldots x_n)}$ we see

that
$$P_\pi(x_1 \ldots x_n)\; a_\delta(x_1 \ldots x_n) = \sum_{\lambda \vdash n} \omega_\lambda(\pi)\, a_{\lambda+\delta}(x_1 \ldots x_n)$$

and we can characterize $\omega_\lambda(\pi)$ as the coefficient of $x^{\lambda+\delta}$ in $\left[P_\pi(x_1 \ldots x_n)\, a_\delta(x_1 \ldots x_n)\right]$.

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Notations: For each polynomial $F = F(x_1 \ldots x_n)$ and each composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $n$, write
$$[x^\alpha]\, F(x)$$
for the coefficient of $x^\alpha$ in $F(x)$.

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We will prove that
$$\chi_{M^\lambda}(\pi) = [x^\lambda]\, P_\pi(x)$$
and
$$\chi_{S^\lambda}(\pi) = [x^{\lambda+\delta}]\left(P_\pi(x)\, a_\delta(x)\right).$$

The latter is known as Frobenius character formula.