The ring of representations and the ring of symmetric functions

For all $n \geq 1$, let R_n be the free abelian group on the isomorphism classes of irreducible representations of S_n. An element of R_n is a finite linear combination

$$[V] = \sum_{\lambda \vdash n} m_\lambda [S^\lambda]$$

of isomorphism classes of Specht module with integer coefficients. If $m_\lambda \geq 0 \; \forall \lambda \vdash n$, then $[V]$ is actually the isomorphism class of a representation $(V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus m_\lambda})$.
When some m_λ's are negative, then V is only a "virtual representation".

We regard R_n as an abelian group, and we ignore its ring structure.
[R_n has a natural multiplication:

$$\otimes : R_n \times R_n \to R_n, \; ([V],[W]) \mapsto [V \otimes W]$$

given by the tensor product].

For $n=0$, define $R_0 = \{$ trivial representation of $S_0 = \{1\}\}$.

Consider the direct sum

$$R = \bigoplus_{n \geq 0} R_n$$

(generated by the virtual representations of all symmetric groups...). We define a product in R by

$$\circ : R_n \times R_m \longrightarrow R_{n+m}$$

$$([V],[W]) \longmapsto [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (V \times W)].$$

notice that if V is a virtual representation of S_n and 2
W is a virtual repr. of S_m, then $[V]\circ[W]$ is the isomorphism class of a virtual representation of S_{n+m}.

We first construct the tensor product of V and W, which is a representation of $S_n \times S_m$ in the obvious way:

$$(\rho_V \otimes \rho_W)(g_1, g_2)(v \otimes w) = \rho_V(g_1)v \otimes \rho_V(g_2)w.$$

Then we identify $S_n \times S_m$ with a subgroup of S_{n+m} (with S_n acting on $\{1 \dots n\}$ and S_m acting on $\{n+1, n+2 \dots n+m\}$) and we induce $V \otimes W$ from $S_n \times S_m$ to S_{n+m}].

It can be checked that $\circ$ is well defined, abelian and associative. So R becomes a graded ring (with unity).

We will show that $R \otimes \mathbb{Q}$ is isomorphic to the graded ring $\Lambda_{\mathbb{Q}}$ of symmetric functions ($\Lambda_{\mathbb{Q}} = \bigoplus_{n \geq 0} \Lambda^n_{\mathbb{Q}}$, with $\Lambda^n_{\mathbb{Q}} = \underline{\text{symmetric functions of degree } n}$).

= symmetric functions with coefficients in $\mathbb{Q}$

With this in mind, we define a map

$$ch^n : R_n \otimes \mathbb{Q} \to \Lambda^n_{\mathbb{Z}} \otimes \mathbb{Q} = \Lambda^n_{\mathbb{Q}}$$

by $$ch^n([V]) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(\mu) P_\mu.$$

If V is a virtual representation, we denote by χ_V its virtual character ($V = \bigoplus \underbrace{m_\lambda}_{\text{in } \mathbb{Q}} S^\lambda \Rightarrow \chi_V = \sum m_\lambda \chi_{S^\lambda}$).

Every partition μ of S_n represents a conjugacy class (take all permutations with cycle-structure μ), and $\chi_v(\mu)$ is the value of the virtual character χ_v on the conjugacy class μ.

We also notice that P_μ is the power sum symmetric function associated to μ, and that z_μ is a constant s.t.

$$z_\mu = \langle P_\mu, P_\mu \rangle = \frac{n!}{\text{cardinality of the c.c. } \mu}.$$

[More precisely, if $\mu = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots, n^{m_n})$ then

$$z_\mu = 1^{m_1} \cdot 2^{m_2} \cdots n^{m_n} \cdot m_1! \cdot m_2! \cdots m_n!$$

Here m_j is the number of parts of μ of length j].

In order to prove that the characteristic map ch^n is invertible, we show that

$$ch^n([S^\lambda]) = s_\lambda.$$

[The result will follow from the fact that the isomorphism classes of Specht modules are a basis for $R_n \otimes \mathbb{Q}$, and the Schur functions are a basis for $\Lambda^n_{\mathbb{Q}}$]. For every partition $\lambda \vdash n$, we can write

$$\bullet \; s_\lambda \underset{\substack{\uparrow \\ \text{because } \left\{\frac{P_\mu}{\sqrt{z_\mu}}\right\} \text{ is an} \\ \text{orthonormal basis}}}{=} \sum_{\mu \vdash n} \langle s_\lambda, \frac{P_\mu}{\sqrt{z_\mu}} \rangle \frac{P_\mu}{\sqrt{z_\mu}} = \sum_{\mu \vdash n} \langle s_\lambda, P_\mu \rangle \frac{P_\mu}{z_\mu}.$$

Let's compute the scalar product $\langle s_\lambda, P_\mu \rangle$.

By Frobenius character formula:

$$P_\mu = \sum_{\lambda \vdash n} \underset{\substack{\uparrow \\ \text{Specht} \\ \text{module}}}{\chi_{S^\lambda}(\mu)} \underset{\substack{\uparrow \\ \text{Shur functions}}}{s_\lambda}.$$

But the Shur functions are an orthonormal basis,

so $\langle P_\mu, s_\lambda \rangle = \chi_{S^\lambda}(\mu)$.

$$\Rightarrow \quad s_\lambda = \sum_{\mu \vdash n} \chi_{S^\lambda}(\mu) \frac{P_\mu}{z_\mu} = ch^n([S^\lambda]), \text{ as claimed!}$$

Remark - Notice that we can equivalently define the characteristic map ch^n as the unique map

$$R_n \otimes \mathbb{Q} \longrightarrow \Lambda^n_{\mathbb{Q}}$$

that carries $[S^\lambda]$ into s_λ, $\forall \lambda \vdash n$.

This definition of ch^n is much more natural!!!

Theorem - ch^n is an isometry.

Proof - For every pair V, W of virtual representations of S_n we compare $\underbrace{\langle \chi_V, \chi_W \rangle}_{\substack{\text{inner product of} \\ \text{class functions on } S_n}}$ with $\underbrace{\langle ch^n[V], ch^n[W] \rangle}_{\substack{\text{inner product of} \\ \text{symmetric functions}}}$.

[We have identified R_n with the group of virtual characters of S_n].

$$\langle ch^n[V], ch^n[W] \rangle = \left\langle \left(\sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(\mu) P_\mu \right), \left(\sum_{\theta \vdash n} \frac{1}{z_\theta} \chi_W(\theta) P_\theta \right) \right\rangle =$$

$$\sum_{\mu,\theta\vdash n} \frac{1}{z_\mu}\frac{1}{z_\theta}\chi_v(\mu)\chi_w(\theta)\underbrace{\langle P_\mu, P_\theta\rangle}_{\overset{\shortparallel}{z_\mu \delta_{\mu,\theta}}} =$$

(because the functions $\left\{\frac{P_\lambda}{\sqrt{z_\lambda}}\right\}$ are orthonormal)

$$= \sum_{\mu\vdash n} \frac{1}{z_\mu}\frac{1}{\cancel{z_\mu}}\chi_v(\mu)\chi_w(\mu)\cancel{z_\mu} =$$

$$= \frac{1}{n!}\sum_{\mu\vdash n} \frac{n!}{z_\mu}\chi_v(\mu)\chi_w(\mu) =$$

size of the c.c of S_n parametrized by μ

because every $\sigma\in S_n$ is conjugate to σ^{-1}, every character takes real values..

$$= \frac{1}{n!}\sum_{\mu\vdash n}\left[\frac{n!}{z_\mu}\right]\chi_v(\mu)\overline{\chi_w(\mu)} =$$

$$= \langle \chi_v, \chi_w\rangle . \checkmark$$

So ch^n is an isometry. ▨

So far we have proved that $\forall n\geq 0$ the map

$$ch^n : R_n\otimes\mathbb{Q} \longrightarrow \Lambda^n_{\mathbb{Q}}$$

is an invertible map, and also an isometry.

Let $\boxed{ch} = \bigoplus_{n\geq 0} ch^n : R\otimes\mathbb{Q} = \left(\bigoplus_{n\geq 0} R_n\otimes\mathbb{Q}\right) \longrightarrow \Lambda_{\mathbb{Q}}$ be the direct sum of all these maps. We call $\boxed{ch}$ ✓ <u>the characteristic map</u>.

We will show that $\boxed{ch}$ is <u>a ring isomorphism</u>, i.e. it also respects multiplication.

We need to introduce some notations.
Let $G = S_n$, for some n. A class function $G \to \Lambda_{\mathbb{Q}}$ are is any mapping from G to $\Lambda_{\mathbb{Q}}$ which is invariant on the conjugacy classes of G.

Examples:

- $p : G \to \Lambda_{\mathbb{Q}}$, $g \mapsto p_\pi$
  where π is the partition that parametrizes the conjugacy class of g in G.

- $\chi : G \to \Lambda_{\mathbb{Q}}$, $g \mapsto \chi(g)$
  for every virtual character χ of G.

[The latter example requires some explanation.
By definition, $\Lambda_{\mathbb{Q}} = \bigoplus_{n \geq 0} \Lambda^n_{\mathbb{Q}}$ and $\Lambda^0_{\mathbb{Q}} = \mathbb{Q}$, so $\mathbb{Q} \subseteq \Lambda_{\mathbb{Q}}$.
The characters of $G = S_n$ have rational values.
This can be deduced from Frobenius character formula:

$$p_\pi = \sum_{\lambda \vdash n} \chi_{S^\lambda}(\pi)\, s_\lambda$$

and from the fact that p_π The Shur functions are a basis of the vector space $\Lambda^n_{\mathbb{Q}}$ (over $\mathbb{Q}$).

So every virtual character χ in $R_n \otimes \mathbb{Q}$ (= a linear combination of chars of Specht modules with rational coefficients) takes values in $\mathbb{Q} \subseteq \Lambda_{\mathbb{Q}}$.
$\Rightarrow$ $\chi : G \to \Lambda_{\mathbb{Q}}$ is well defined.]

If ψ_1, ψ_2 are any two class functions $G \to \Lambda_{\mathbb{Q}}$, we define a product $\langle \psi_1, \psi_2 \rangle$ by:

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_1(g)\, \psi_2(g^{-1})$$

This is an element of $\Lambda_{\mathbb{Q}}$ (because $\Lambda_{\mathbb{Q}}$ is a ring)

$$= \frac{1}{n!} \sum_{\mu \vdash n} \underbrace{\frac{n!}{z_\mu}}_{\text{size of the c.c.} \leftrightarrow \mu} \psi_1(\mu)\, \psi_2(\mu) =$$

g and g^{-1} belong to the same conjugacy class

recall that $G = S_n$ and conjugacy classes of S_n are parametrized by partitions

$$= \sum_{\mu \vdash n} \frac{1}{z_\mu} \psi_1(\mu)\, \psi_2(\mu).$$

- In particular, if $\psi_1 = \chi$ and $\psi_2 = p$, we find that

$$\langle \chi, p \rangle = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi(\mu)\, p_\mu = ch(\chi).$$

If H is a subgroup of G, we define a notion of induction (and restriction) of class functions $H \to \Lambda_{\mathbb{Q}}$ (or $G \to \Lambda_{\mathbb{Q}}$).

- [RESTRICTION] If $H \leq G$, and $\psi : G \to \Lambda_{\mathbb{Q}}$ is a class function on G, define

$$\psi\downarrow_H : H \to \Lambda_{\mathbb{Q}}, \quad h \mapsto \psi(h).$$

INDUCTION] If $H \leq G$, and $\varphi : H \to \Lambda_{\mathbb{Q}}$ is a class function on H, we define

$$\varphi\uparrow^G : G \to \Lambda_{\mathbb{Q}},\ g \mapsto \frac{1}{|H|} \sum_{x \in G} \varphi(x^{-1} g x)$$

where φ is intended to be 0 out of H, so $\varphi(x^{-1}gx) = 0$ if $x^{-1}gx \notin H$.

Notice that these notion of induction and restriction of class functions $G \to \Lambda_{\mathbb{Q}}$ (or $H \to \Lambda_{\mathbb{Q}}$) are the natural generalization of the notions of induction and restriction of characters (which are class functions $G \to \mathbb{Q}$).

It can be shown that Frobenius reciprocity still holds:

$$\langle \varphi\uparrow^G, \psi \rangle = \langle \varphi, \psi\downarrow_H \rangle$$

for every class functions $\varphi : H \to \Lambda_{\mathbb{Q}}$, $\psi : G \to \Lambda_{\mathbb{Q}}$.

[Of course you could replace $\Lambda_{\mathbb{Q}}$ with any graded ring over $\mathbb{Q}$].

With this machinery it's easy to prove that $\boxed{ch}$ is a ring isomorphism.

Theorem – The characteristic map preserves the product: for every virtual character χ of S_n and ψ of S_n, $ch(\chi \circ \psi) = ch(\chi)\, ch(\psi)$.

proof- $\mathrm{ch}(\chi \circ \psi) = \mathrm{ch}\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} \chi \otimes \psi\right) \underset{\text{remark: } \mathrm{ch} = \langle \circ, p \rangle}{=}$

$$= \left\langle \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} \chi \otimes \psi, \ p \right\rangle \underset{\text{Frobenius reciprocity}}{=}$$

$$= \left\langle \chi \otimes \psi, \ p\downarrow_{S_n \times S_m} \right\rangle \underset{\text{def. of } \langle , \rangle}{=}$$

$$= \frac{1}{|S_n \times S_m|} \sum_{\pi\sigma \in S_n \times S_m} (\chi \otimes \psi)(\pi\sigma) \ p_{\pi\sigma} =$$

$$= \frac{1}{n!\,m!} \sum_{\substack{\pi \in S_n \\ \sigma \in S_m}} \chi(\pi)\, \psi(\sigma)\, p_\pi\, p_\sigma =$$

$$= \left[\frac{1}{n!} \sum_{\pi \in S_n} \chi(\pi)\, p_\pi\right]\left[\frac{1}{m!} \sum_{\sigma \in S_m} \psi(\sigma)\, p_\sigma\right] =$$

$$= \langle \chi, p \rangle \langle \psi, p \rangle =$$

$$= \mathrm{ch}(\chi) \cdot \mathrm{ch}(\psi). \quad \blacksquare$$