Problem: Complete the character Table. Then compute the sizes and the order of the elements in each conjugacy class.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1					
χ_2	1	i	1		-1
χ_3		0	-1	0	0
χ_4	1		1	-1	1
χ_5	1	$-i$	1		-1

- The Trivial representation is irreducible $\Rightarrow$ one of the χ_i's must be the Trivial character. It's clearly χ_1.

- $\overline{\chi_2}$ is an irreducible character. It must be χ_5. Notice that $\chi_2(\Gamma_4)$ is a root of unity. Call it x.

Also set $y = \chi_3(\Gamma_1)$ and $z = \chi_4(\Gamma_2)$.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1	1	1	1	1	1
χ_2	1	i	1	x	-1
χ_3	y	0	-1	0	0
χ_4	1	z	1	-1	1
χ_5	1	$-i$	1	$\overline{x}$	-1

We can complete the Table using the orthogonality of the columns.

- 1st and 3rd column $\rightarrow \displaystyle\sum_{j=1}^{5} \overline{\chi_j(\Gamma_1)}\,\chi_j(\Gamma_3) = 0 \iff 1+1-\overline{y}+1+1=0$

$$\iff \overline{y} = 4 \iff y = 4.$$

- 2nd and 3rd column $\rightarrow \displaystyle\sum_{j=1}^{5} \overline{\chi_j(\Gamma_2)}\,\chi_j(\Gamma_3) = 0 \iff 1-i+\overline{z}+i=0$

$$\iff \overline{z} = -1 \iff z = -1.$$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1	1	1	1	1	1
χ_2	1	i	1	x	-1
χ_3	4	0	-1	0	0
χ_4	1	-1	1	-1	1
χ_5	1	$-i$	1	$\bar{x}$	-1

We just need to find x.

the easiest way to do so is to observe that if $g \in \Gamma_2$ then $g^{-1} \in \Gamma_4$. Indeed $g^{-1} \notin \Gamma_2$ (or $\chi_2(\Gamma_2)$ would be real), and $g^{-1} \notin \Gamma_1$ or Γ_3 or Γ_5 or $\chi_2(\Gamma_{1,3,5}) = \overline{\chi_2(\Gamma_2)} = -i$.

So for all $j = 1..5$, $\chi_j(\Gamma_4) = \overline{\chi_j(\Gamma_2)}$.

In particular, $x = -i$ and $\bar{x} = i$.

The complete character Table is:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1	1	1	1	1	1
χ_2	1	i	1	$-i$	-1
χ_3	4	0	-1	0	0
χ_4	1	-1	1	-1	1
χ_5	1	$-i$	1	i	-1

We find:

$$|G| = \sum_{j=1}^{5} \chi_j(\Gamma_1)^2 = 1+1+16+1+1 = 20$$ (because $\Gamma_1 = \{1\}$).

The sizes of the conjugacy classes can be found using the formula:

$$\sum_{j=1}^{5} |\chi_j(\Gamma)|^2 = \frac{|G|}{\text{size of } \Gamma}.$$

We get:

- size of $\Gamma_1 = 1$
- size of $\Gamma_2 = \dfrac{20}{1+1+1+1} = 5 =$ size of Γ_4 = size of Γ_5
- size of $\Gamma_3 = \dfrac{20}{1+1+1+1+1} = 4$.

$[\, 1+5+5+5+4 = 20 \checkmark \,]$

Finally, we find the orders of the elements in each conjugacy class. Notice that the stabilizers have orders 20, 4, 5, 4, 4. Therefore:

- elements in Γ_3 have order 5.
- elements in Γ_2 and Γ_4 have order 4 (The order cannot be 2 because $\chi_2(\Gamma_2)$ and $\chi_2(\Gamma_4)$ are not 2-nd rooth of unity, and χ_2 is a linear character).

Because Γ_1 is the identity (order 1) and because every group of order 20 contains an element of order 2, we conclude that the elements in Γ_5 have order 2.

PROBLEM

Examine The following character Table:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	1	1	ω	ω^2	1	ω	ω^2
χ_4	1	1	ω^2	ω	1	ω^2	ω
χ_5	1	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_6	1	1	ω^2	ω	-1	$-\omega^2$	$-\omega$
χ_7	6	-1	0	0	0	0	0

and answer the following questions:

1. What is The size of G?
2. What are The sizes of The conjugacy classes?
3. What are The orders of The Kernels of each irreducible representation?
4. Which irreducible representation is faithful?
5. Find The orders of The elements in each conjugacy class.
6. Find all normal subgroups of G. Identify $Z(G)$ and $[G,G]$.

① $|G| = \sum_{i=1}^{7} \chi_i(\Gamma_1)^2 = 1+1+1+1+1+1+36 = 42$

[We use the fact that $\Gamma_1 = \{1\}$ because it's the only conjugacy class on which every character has a positive integer value].

② Clearly Γ_1 has size 1, because $\Gamma_1 = \{1\}$.
To find the size of the other conjugacy classes, we use the formula:

$$\sum_{j=1}^{7} |\chi_j(\Gamma)|^2 = \frac{|G|}{\text{size of } \Gamma}$$

Notice that ω and ω^2 are values of linear characters, so they are roots of unity and therefore have modulus 1.
We get:

- size of $\Gamma_2 = \frac{42}{7} = 6$
- size of $\Gamma_j = \frac{42}{6} = 7 \quad \forall j = 3, 4, 5, 6, 7$.

③ Recall that $\operatorname{Ker} \rho = \operatorname{Ker} \chi_\rho = \{ g \in G : \chi_\rho(g) = \chi_\rho(1) \}$.
For all $i = 1..7$, let ρ_i be the irreducible representation of G with character ρ_i. Then:

- $\operatorname{Ker} \rho_1 = G$ (order 42)
- $\operatorname{Ker} \rho_2 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (order $1+6+7+7 = 21$)
- $\operatorname{Ker} \rho_3 = \operatorname{Ker} \rho_4 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_5$ (order $1+6+7 = 14$)
- $\operatorname{Ker} \rho_5 = \operatorname{Ker} \rho_6 = \Gamma_1 \cup \Gamma_2$ (order $1+6 = 7$)
- $\operatorname{Ker} \rho_7 = \Gamma_1$ (order 1).

(4) Only ρ_7 is faithful, because $\operatorname{Ker}(\rho_7) = \{1\}$.

(5) Clearly, the order of the element in Γ_1 is 1, because $\Gamma_1 = \{1\}$. For $j \geq 2$, we look at the sizes of the stabilizer of an element in Γ_j $\left(= \frac{|G|}{\text{size of } \Gamma_j}\right)$.

Sizes of stabilizers : $\underset{\Gamma_2}{7}, \underset{\Gamma_3}{6}, \underset{\Gamma_4}{6}, \underset{\Gamma_5}{6}, \underset{\Gamma_6}{6}, \underset{\Gamma_7}{6}$.

Because 7 is a prime number, every element of Γ_2 must have order 7.

Because $3 \mid 42 = |G|$, G must have an element of order 3. If such an element belongs to Γ_j, then $\chi_i(\Gamma_j)$ must be a cubic root of unity, for all $i = 2 .. 6$ (because $\chi_2, \chi_3 .. \chi_6$ are linear characters).

So only Γ_3 or Γ_4 can contain an element of order 3.

It immediately follows that:

- w is a cubic-root of unity
- $w^2 = \bar{w}$
- $\chi_i(\Gamma_3) = \overline{\chi_i(\Gamma_4)}$ $\forall i = 1 .. 7$.

For every $g \in \Gamma_3$, g has order 3 and $g^{-1} \in \Gamma_4$ has also order 3.

$\Rightarrow$ the elements of Γ_3 and Γ_4 have order 3.

Next, we notice that G must contain an element of order 2 (because $2 \mid |G| = 42$) and such an element must lie in Γ_5. Elements of Γ_6 and Γ_7 must have order 6, since They are included in a group of order 6 and Their order is neither 1, nor 2 nor 3).

⑥ Finally, we look for The normal subgroups of G.
We already know That

$$H_1 = \{1\} \qquad \text{(order 1)}$$

$$H_2 = \Gamma_1 \cup \Gamma_2 \qquad \text{(order 7)}$$

$$H_3 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 \qquad \text{(order 14)}$$

$$H_4 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \qquad \text{(order 21)}$$

$$H_5 = G \qquad \text{(order 42)}$$

are normal subgroups, because They are Kernel of representations.
Moreover, we know That The conjugacy classes have sizes : 1, 6, 7, 7, 7, 7, 7.

Suppose $H \trianglelefteq G \Rightarrow |H| \mid 42$ and H is a union of conjugacy classes, one of which is always $\Gamma_1 = \{1\}$.
The divisors of 42 are 1, 2, 3, 6, 7, 14, 21, 42.
Clearly H_1 is The only (normal) subgroup of order 1, H_5 is the only (normal) subgroup of order 42 and There are no subgroups of order 2, 3, or 6.
If $\# H = 7$, then H can only contain $\{1\}$ and the conjugacy class whose elements have order 7, so $H = \Gamma_1 \sqcup \Gamma_2 = H$

If $|H| = 14$, Then H must contain an element of order 1, an element of order 2 and an element of order 7, so H must be $= \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 = H_3$.

If $|H| = 21$, Then H must contain an element of order 7 and an element of order 3. Recall The inverses of The elements in Γ_3 lie in Γ_4, and viceversa.

So $$H = \underset{\text{order } 1}{\Gamma_1} \cup \underset{\text{order } 7}{\Gamma_2} \cup \underbrace{\Gamma_3 \cup \Gamma_4}_{\text{order } 3, \text{ with Their inverses} \ldots} = H_4.$$

So H_1, H_2, H_3, H_4, H_5 are the <u>only</u> normal subgroups of G.

$$\begin{tikzcd}
 & H_5 = G \arrow[dl, dash] \arrow[dr, dash] & \\
H_3 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 \arrow[dr, dash] & & H_4 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \arrow[dl, dash] \\
 & H_2 = \Gamma_1 \cup \Gamma_2 \arrow[d, dash] & \\
 & H_1 = \{1\} &
\end{tikzcd}$$

We also notice That

- $Z(G) = \{1\} = H_1$ (because Γ_1 is The only c.c. of size 1)
- $[G,G] = \bigcap_{\chi \text{ linear}} \ker \chi = \bigcap_{i=1}^{6} \ker \chi_i = \Gamma_1 \cup \Gamma_2 = H_2$.