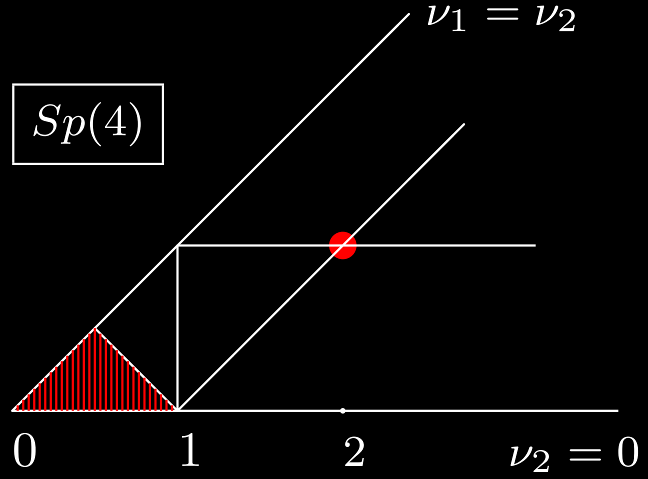
Subquotients of Minimal Principal Series



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July 2008

PART 1

Introduction

Preliminary Definitions and Notation

Langlands Quotients of Minimal Principal Series

- **Notation**
$$\begin{cases} G & \text{real split group} \\ K \subset G & \text{maximal compact subgroup} \\ H = MA & \text{Cartan subgroup; } M = K \cap H \simeq (\mathbb{Z}_2)^l \\ P = MAN & \text{minimal parabolic subgroup} \end{cases}$$

- **Parameters**
$$\begin{cases} (\delta, V^\delta) \in \widehat{M} & \\ \nu \in \widehat{A} \simeq \mathfrak{a}^* & \text{real and weakly dominant} \end{cases}$$

- **Principal Series** $\boxed{X(\delta, \nu)} = \boxed{\mathrm{Ind}_{P=MAN}^{G}(\delta \otimes \nu \otimes triv)}$

  Note that $\forall\, \mu \in \widehat{K}$, $\mathrm{mult}(\mu, X(\delta,\nu)|_K) = \mathrm{mult}(\delta, \mu|_M)$.

- **Langlands Quotient:** $L(\delta, \nu)$ **is the largest completely reducible quotient of** $X(\delta, \nu)$**.**

The problem

Which irreducible components of $L(\delta, \nu)$ are (not) unitary?

We illustrate some techinques to relate the unitarizability of (the irreducible components of) $L(\delta, \nu)$ with the quasi-spherical unitary dual for certain extended Hecke algebras.

The main tool is a generalization of Barbasch's notion of "petite" K-types for *spherical* principal series.

PART 2

On the reducibility of $L(\delta, \nu)$

(fine K-types, good roots, R-groups...)

Fine K-types

For each root α we choose a Lie algebra homomorphism

$$\phi_\alpha \colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0$$

and we define $G_\alpha \simeq SL(2,\mathbb{R})$ to be the connected subgroup of G with Lie algebra $\phi_\alpha(\mathfrak{sl}(2,\mathbb{R}))$. Let $K_\alpha \subset G_\alpha$ be the corresponding $SO(2)$-subgroup.

A K-type $\mu \in \widehat{K}$ is called **fine** if the restriction of μ to K_α only contains the representations 0, 1 and -1 of $SO(2)$.

For each $\delta \in \widehat{M}$, there is a fine K-type μ_δ containing δ.

$$\mu_\delta|_M = \bigoplus_{\pi \in W\text{-}orbit\ of\ \delta} \pi$$

Note that $\mathrm{mult}(\delta, \mu_\delta) = 1$, so $\mathrm{mult}(\mu_\delta, X(\delta,\nu)) = 1$.

Irreducible Components of $L(\delta, \nu)$

Fix $\delta \in \widehat{M}$, and set $A(\delta) = \{\text{fine } K\text{-types containing } \delta\}$.
For all $\mu_\delta \in A(\delta)$ define:

$L(\delta,\nu)(\mu_\delta)$=unique irreducible subquotient of $X(\delta,\nu)$ containing μ_δ

Then

- *$L(\delta,\nu)(\mu_\delta)$ is well defined, because mult$(\mu_\delta, X(\delta,\nu)) = 1$.*
- *$L(\delta,\nu)(\mu_\delta)$ may contain other fine K-types (other than μ_δ).*
- *$L(\delta,\nu) = \sum_{\pi \in A(\delta)} L(\delta,\nu)(\pi)$. Hence the irreducible components of the Langlands quotient $L(\delta,\nu)$ are precisely the irreducible subquotients $\{L(\delta,\nu)(\pi)\}_{\pi \in A(\delta)}$.*

Note: # of distinct irreducible subquotients $= \# R_\delta(\nu)$.

The good roots for δ

Fix $\delta \in \widehat{M}$. For each root α, choose a Lie algebra homomorphism $\phi_\alpha \colon \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{g}$. Set

$$Z_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

- Z_α is a generator for $\mathrm{Lie}(K_\alpha) \simeq \mathfrak{so}(2)$
- $\sigma_\alpha = \exp(\pi Z_\alpha/2)$ is a representative in K for the reflection s_α
- $m_\alpha = \sigma_\alpha^2$ belongs to M, and $m_\alpha^2 = 1$.

Definition A root α is "***good for*** $\boldsymbol{\delta}$" if $\delta(m_\alpha) = 1$.

The R-group for δ

Fix an M-type δ, and a (real) character ν of A.

Define:
- $\Delta_\delta = \{\alpha : \delta(m_\alpha) = 1\}$ — the good roots for δ (a root system)
- $W_\delta < W$ — the stabilizer of δ $(\sigma\delta(m) = \delta(\sigma^{-1} m \sigma))$
- W_δ^0 — the Weyl group of Δ_δ $(W_\delta^0 \trianglelefteq W_\delta)$
- $R_\delta \equiv \frac{W_\delta}{W_\delta^0}$ — the R-group of δ (G split $\Rightarrow R_\delta$ abelian)

Also set:

$$R_\delta(\nu) = \frac{\{w \in W_\delta : w\nu = \nu\}}{\{w \in W_\delta^0 : w\nu = \nu\}}$$

Then

$$\#\text{ fine } K\text{-types containing } \delta = \# R_\delta$$

and

$$\#\text{ irreducible subquotients } = \# R_\delta(\nu).$$

R-groups and Langlands subquotients

$\widehat{R_\delta}$ acts simply transitively on the set of *fine* K-types containing δ. Fix a fine K-type $\mu_\delta \in A(\delta)$. There is a bijective correspondence

$$\boxed{A(\delta) = \{\text{fine } K\text{-types containing } \delta\} \Leftrightarrow \widehat{R_\delta}}.$$

Then $\quad \#\text{ fine } K\text{-types containing } \delta = \#\,\widehat{R_\delta} \overset{R_\delta\ abel.}{=} \#\,R_\delta.$

Two fine K-types are in the same Langlands subquotient iff they lie in the same orbit of $R_\delta(\nu)^\perp$, where $R_\delta(\nu)^\perp = \{\chi \in \widehat{R_\delta} : \chi|_{R_\delta(\nu)} = 1\}$.

$$\boxed{\{\text{irred. subquotients}\} \Leftrightarrow \{\text{orbits of } R_\delta(\nu)^\perp\} \Leftrightarrow \frac{\widehat{R_\delta}}{R_\delta(\nu)^\perp} \Leftrightarrow \widehat{R_\delta(\nu)}}$$

Then $\quad \#\text{ irreducible subquotients} = \#\,\widehat{R_\delta(\nu)} \overset{R_\delta(\nu)\ abel.}{=} \#\,R_\delta(\nu).$

Some examples

Let $\boxed{G = SL(2)}$. Then

- $K{=}SO(2)$, $M{=}\{\pm I\} \simeq (\mathbb{Z}_2)$ and $\widehat{M}{=}\{triv,\, sign\}$
- $\Delta^+ = \{\alpha\}$ and $m_\alpha = -I$. So $triv(m_\alpha) = +1$, $sign(m_\alpha) = -1$.

δ	α	Δ_δ	W_δ^0	W_δ	R_δ	$R_\delta(\nu)$
triv	*good*	$\{\pm\alpha\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$
sign	*bad*	$\emptyset$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{1\}$ if $\nu > 0$, $\mathbb{Z}_2$ if $\nu = 0$.

- If $\delta{=}triv$, then there is a unique fine K-type containing δ $(\pi = 0)$ and a unique irreducible subquotient.
- If $\delta{=}sign$, then there are two fine K-types containing δ $(\pi = 1, -1)$. There is a unique irreducible subquotient if $\nu > 0$, and two irreducible subquotients if $\nu = 0$.

Let $\boxed{G = Sp(4,\mathbb{R})}$. Then

- $K{=}U(2)$, $M{=}\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\} \simeq \mathbb{Z}_2{\times}\mathbb{Z}_2$ and $\widehat{M}{=}\{\delta_{+,+};\, \delta_{+,-};\, \delta_{-,+};\, \delta_{-,-}\}$
- $\Delta^+ = \{\epsilon_1 \pm \epsilon_2,\ 2\epsilon_1,\ 2\epsilon_2\}$.
- $m_{\epsilon_1 \pm \epsilon_2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; $m_{2\epsilon_1} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$; $m_{2\epsilon_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Every representation of M is stable under any sign change (because $\sigma_{2\epsilon_1}$ and $\sigma_{2\epsilon_2}$ are diagonal).

δ	W^0_δ	W_δ	R_δ	$R_\delta(\nu)$
$\delta_{+,-}$	$\langle s_{2\epsilon_1} \rangle$	$\langle s_{2\epsilon_1}, s_{2\epsilon_2} \rangle$	$\mathbb{Z}_2 = \langle s_{2\epsilon_2} \rangle$	$\{1\}$ if $a_n \neq 0$, $\mathbb{Z}_2$ o.w.
$\delta_{-,-}$	$\langle s_{\epsilon_1 \pm \epsilon_2} \rangle$	$W(C_2)$	$\mathbb{Z}_2 = \langle s_{2\epsilon_2} \rangle$	$\{1\}$ if $a_n \neq 0$, $\mathbb{Z}_2$ o.w.

If $\delta{=}\delta_{+,-}$ or $\delta_{-,-}$, then δ is contained in two fine K-types (μ_δ and μ^*_δ). If $a_n \neq 0$, there is one irreducible subquotient containing both μ_δ and μ^*_δ. If $a_n{=}0$, these fine K-types are apart, and there are 2 subquotients.

Let $\boxed{G = Sp(n)}$. Then

- $K = U(n)$, $M = \{\mathrm{diag}(a_1, \ldots, a_n) : a_k = \pm 1\} \simeq (\mathbb{Z}_2)^n$
- $\Delta^+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\} \cup \{2\epsilon_j : 1 \leq j \leq n\}$
- $m_{\epsilon_i \pm \epsilon_j} = \mathrm{diag}(a_1, \ldots, a_n)$; $a_i = a_j = -1$ and $a_k = +1$ otherwise.
- $m_{2\epsilon_j} = \mathrm{diag}(a_1, \ldots, a_n)$; $a_j = -1$ and $a_k = +1$ otherwise.

For $p = 1 \ldots n$, let $\delta_p = (\underbrace{+, \ldots, +}_{n-p}, \underbrace{-, \ldots, -}_{p})$:

$$\boxed{\delta_p(\mathrm{diag}(a_1, \ldots, a_n)) = \prod_{k=n-p+1}^{n} a_k}$$

- $\alpha = \epsilon_i \pm \epsilon_j$ is good for $\delta_p \Leftrightarrow i < j \leq n - p$, or $n - p < i < j$.
- $\alpha = 2\epsilon_j$ is good for $\delta_p \Leftrightarrow j \leq n - p$.

$\Rightarrow$ The good roots for δ_p form a root system of type $C_{n-p} \times D_p$.

- $W = S_n \lhd (\mathbb{Z}_2)^n$ consists of permutation and sign changes.
- The representation δ_p of M is stable under any sign change, and under permutations of $\{1 \dots n-p\} \cup \{n-p+1 \dots n\}$. Hence $\boxed{W_{\delta_p} = W(C_{n-p} \times C_p).}$
- $\boxed{W^0_{\delta_p} = W(C_{n-p} \times D_p)}$ has index 2 in $W_{\delta_p} \Rightarrow \boxed{R_\delta = \mathbb{Z}_2}$.
  (δ_p is contained in two fine K-types: $\mu_\delta = \Lambda^p(\mathbb{C}^n)$ and its dual.)
- Let $\nu = (a_1, a_2 \dots, a_n)$, with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $R_{\delta_p}(\nu) \neq \{1\} \Leftrightarrow s_{2\epsilon_k}\nu = \nu$ for some $k > n-p \Leftrightarrow a_n = 0$.

$W^0_{\delta_p}$	W_{δ_p}	R_{δ_p}	$R_{\delta_p}(\nu)$
$W(C_{n-p} \times D_p)$	$W(C_{n-p} \times C_p)$	$\mathbb{Z}_2$	$\{1\}$ if $a_n \neq 0$, $\mathbb{Z}_2$ o.w.

If $a_n \neq 0$, there is one irred. subquotient containing both μ_δ and μ^*_δ.
If $a_n = 0$, the 2 fine K-types are apart, and there are 2 subquotients.

PART 3

Unitarity Question

Which irreducible subquotients $L(\delta, \nu)(\pi)$ are unitary?

Hermitian forms on the irreducible subquotients

Assume that ν is weakly dominant. Let $Q \subset G$ be the parabolic defined by ν and let $w \in W$ be a Weyl group element such that

$$wQw^{-1} = \bar{Q}, \quad w\delta \simeq \delta, \quad w\nu = -\nu.$$

Fix a fine K-type μ_δ. One can define an intertwining operator

$$A(w,\delta,\nu)\colon X(\delta,\nu) \to X(\delta,-\nu)$$

(normalized on μ_δ) such that

- $A(w,\delta,\nu)$ has no poles, and $\overline{Im(A(w,\delta,\nu))} = L(\delta,\nu)$.
- The operator $\boxed{\mathcal{A}(w,\delta,\nu) = \mu_\delta(w)A(w,\delta,\nu)}$ is Hermitian, and induces a *non-degenerate* invariant Hermitian form on $L(\delta,\nu)$.
- Every subquotient $L(\delta,\nu)(\pi)$ inherits a Hermitian form.

Unitarity of an (Hermitian) irreducible subquotient

Assume that $L(\delta,\nu)$ is Hermitian.

- The Hermitian form on $L(\delta,\nu)(\pi)$ is induced by an operator $A(w,\delta,\nu)\colon X(\delta,\nu) \to X(\delta,-\nu)$.
- $A(w,\delta,\nu)$ gives rise to an operator $\mathcal{A}_\mu(w,\delta,\nu)$ on $\mathrm{Hom}_K(\mu, X(\delta,\nu))$, for every K-type $\mu \in \widehat{K}$.

$L(\delta,\nu)(\pi)$ is unitary $\Leftrightarrow$ the corresponding block of $\mathcal{A}_\mu(w,\delta,\nu)$ is semidefinite $\forall \mu \in \widehat{K}$

$\Rightarrow$ *To check the unitarity of $L(\delta,\nu)(\pi)$ we need to compute the signature of infinitely many operators $\{\mathcal{A}_\mu(w,\delta,\nu)\}_{\mu\in\widehat{K}}$*

Computations can be reduced to an $SL(2)$-calculation.

Rank-one reduction

- Attach an $SL(2)$-subgroup to each root.
- Using Frobenius reciprocity, interpret $\mathcal{A}_\mu(w, \delta, \nu)$ as an operator on $\mathrm{Hom}_M(\mu, \delta)$.
- Choose a minimal decomposition of w as a product of simple reflections. The operator $\mathcal{A}_\mu(w, \delta, \nu)$ decomposes accordingly:

$$\mathcal{A}_\mu(w, \delta, \nu) = \prod_{\alpha \; simple} \mathcal{A}_\mu(s_\alpha, \rho, \lambda).$$

- The factor associated to a simple reflection α behaves as an operator for the rank-one group $M\,SL(2)_\alpha$.
- Explicit formulas are known for $SL(2)$. So we know how to compute the various α-factors of the operator.

The "α-factor" $\mathcal{A}_\mu(s_\alpha, \rho, \lambda)$

Let α be a simple root, and let ρ be an M-type in the W-orbit of δ.

Let μ_δ be the (fixed) fine K-type used for the normalization. We can assume that both ρ and $s_\alpha\rho$ are realized inside μ_δ.

To construct the operator

$$\mathcal{A}_\mu(s_\alpha, \rho, \lambda) \colon \mathrm{Hom}_M(\mu, \rho) \longrightarrow \mathrm{Hom}_M(\mu, s_\alpha\rho)$$

we look at the restriction of μ to the $SL(2)$ attached to α.

Let Z_α be a generator of the corresponding $\mathfrak{so}(2)$. Consider the action of Z_α^2 on $\mathrm{Hom}_M(\mu, \rho)$ by $T \mapsto T \circ \mathrm{d}\mu(Z_\alpha)^2$, and let

$$\mathrm{Hom}_M(\mu, \rho) = \bigoplus_{l \in \mathbb{N}} E^\alpha(-l^2)$$

be the corresponding decomposition in generalized eigenspaces.

The operator $\mathcal{A}_\mu(s_\alpha, \rho, \lambda)$ acts on $E^\alpha(-l^2)$ by

$$T \mapsto c_l(\alpha, \lambda)\, \mu_\delta(\sigma_\alpha) \circ T \circ \mu(\sigma_\alpha^{-1}).$$

The scalars $c_l(\alpha, \lambda)$

Set $\xi = \langle \lambda, \check{\alpha} \rangle$. For every integer $l \in \mathbb{N}$, we have:

- $$c_{2m}(\alpha, \lambda) = (-1)^m \frac{(1-\xi)(3-\xi)\cdots(2m-1-\xi)}{(1+\xi)(3+\xi)\cdots(2m-1+\xi)}$$

- $$c_{2m+1}(\alpha, \lambda) = (-1)^m \frac{(2-\xi)(4-\xi)\cdots(2m-\xi)}{(2+\xi)(4+\xi)\cdots(2m+\xi)}$$

Note that the scalar $c_l(\alpha, \lambda)$ becomes rather complicated if the eigenvalue l of $\mathrm{d}\mu(iZ_\alpha)$ is big.

PART 4

Petite K-types

Examples and Definition.

The idea of petite K-types

To obtain necessary and sufficient conditions for the unitarity of a Langlands subquotient, we need to study the signature of infinitely many operators $\mathcal{A}_\mu(w, \delta, \nu)$ (one for each $\mu \in \widehat{K}$). Computations are hard if μ is "large".

Alternative plan:

1. Select a small set of "petite" K-types on which computations are easy.
2. Only compute the signature of $\mathcal{A}_\mu(w, \delta, \nu)$ *only* for μ petite, hoping that the calculation will rule out large non-unitarity regions.

This approach will provide necessary conditions for unitarity:

$$L(\delta, \nu)(\pi) \,\text{unitary} \;\Rightarrow\; \mathcal{A}_\mu(w, \delta, \nu) \,\text{pos. semidefinite}, \forall\, \mu \,\text{petite}$$

Petite K-types for real split groups

WISH LIST:

- Petite K-types should form a small set.
- The operators $\{\mathcal{A}_\mu(\delta,\nu) \colon \mu \textit{ petite}\}$ should be "easy" to compute.
- The operators $\{\mathcal{A}_\mu(\delta,\nu) \colon \mu \textit{ petite}\}$ should rule out as many non-unitarity points as possible.

PROBLEM: How do we define "petite" K-types?

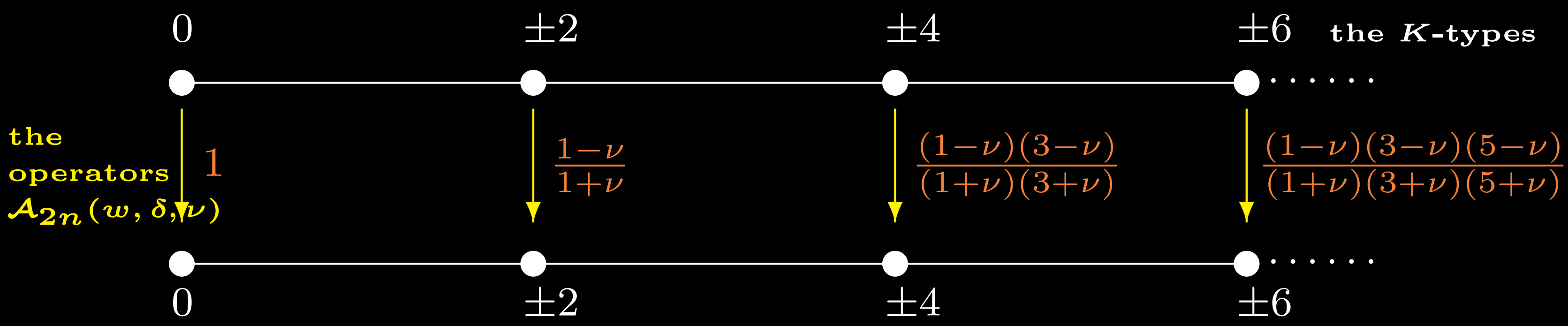
Inspiration comes from the $SL(2,\mathbb{R})$-example.

Spherical Langlands subquotients for $SL(2,\mathbb{R})$

$$G = SL(2,\mathbb{R}),\ K = SO(2,\mathbb{R}),\ \widehat{K} = \mathbb{Z},\ M = \mathbb{Z}_2,\ \delta = trivial,\ \nu > 0$$

There is one operator $\mathcal{A}_{2n}(w,\delta,\nu)$ for every even integer

The domain of $\mathcal{A}_{2n}(w,\delta,\nu)$ is 1-dimensional, so the operator $\mathcal{A}_{2n}(w,\delta,\nu)$ acts by a scalar:

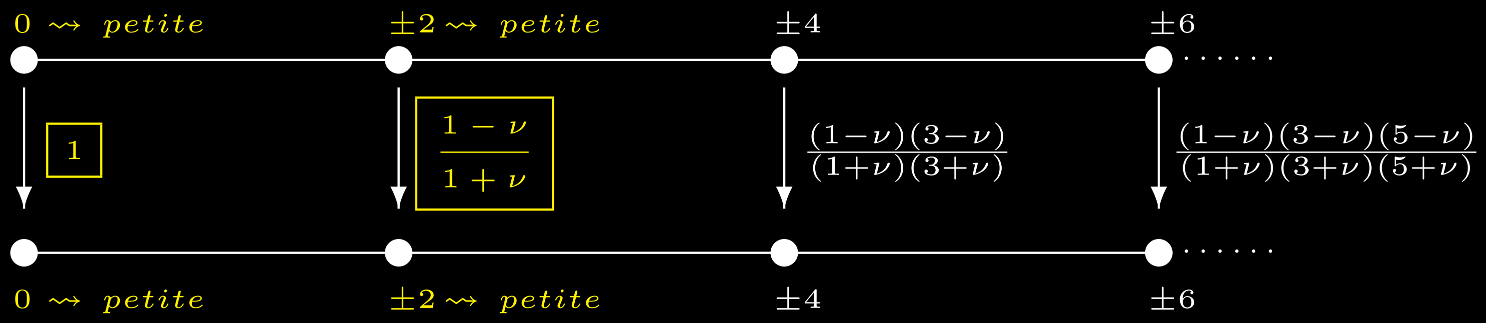


$L(\delta,\nu)$ is unitary $\Leftrightarrow$ $\mathcal{A}_{2n}(w,\delta,\nu) \geq 0, \forall\, n$ $\Leftrightarrow$ $0 < \nu \leq 1$

Note that we could have used $\mu = 0$ and ± 2 alone!

Spherical petite K-types for $SL(2,\mathbb{R})$

There are 3 spherical ***petite*** K-types: $\mu = 0$, $\mu = 2$ and $\mu = -2$.
The corresponding operators $\mathcal{A}_\mu(w, \delta, \nu)$ are:



Note that:

- The K-types $\{\mu = 0, \pm 2\}$ form a small set. ✓
- The operators $\{\mathcal{A}_\mu(w, \delta, \nu) \colon \mu = 0, \pm 2\}$ are "easy". ✓
- The operators $\{\mathcal{A}_\mu(w, \delta, \nu) \colon \mu = 0, \pm 2\}$ rule out **all** the non-unitarity points of $L(triv, \nu)$. ✓

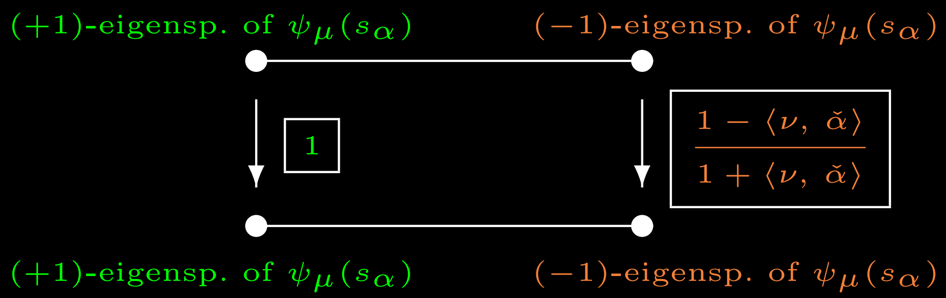
So these K-types have all the desired properties (and more!).

What is special about the K-types 0, ± 2?

$\forall \mu \in \widehat{K}$, $\mathcal{A}_\mu(w, triv, \nu)$ is an operator on $\mathrm{Hom}_M(\mu, triv) = (V_\mu^*)^M$.
This space carries a representation ψ_μ of W. If $\mu = 0, \pm 2$, we have:

μ	the W-type ψ_μ on $(V_\mu^*)^M$	(+1)-eigenspace of s_α	(-1)-eigenspace of s_α	$\mathcal{A}_\mu(w, triv, \nu)$
0	$triv$	$(V_\mu^*)^M$	$\{0\}$	1
± 2	$sign$	$\{0\}$	$(V_\mu^*)^M$	$\frac{1-\langle \nu, \check{\alpha} \rangle}{1-\langle \nu, \check{\alpha} \rangle}$

In both cases $\mathcal{A}_\mu(w, triv, \nu) = \mathcal{A}_\mu(s_\alpha, triv, \nu)$ acts by:



It behaves like an operator for an affine graded Hecke algebra!

Affine graded Hecke algebras

To every real split group G, we associate an affine graded Hecke algebra as follows. Let $\mathfrak{h}$ be the complexification of the Cartan, and let $\mathbb{A} = S(\mathfrak{h})$. The **affine graded Hecke algebra** associated to G is the vector space

$$\mathbb{H} := \mathbb{C}[W] \otimes \mathbb{A}$$

with commutator relations:

$$x t_{s_\alpha} = t_{s_\alpha} s_\alpha(x) + \langle x, \alpha \rangle \qquad \forall \alpha \in \Pi,\ x \in \mathfrak{h}.$$

For all $\nu \in \mathfrak{a}^*$, one defines the principal series $\boldsymbol{X^{\mathbb{H}}(\nu) :=} \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_{\boldsymbol{\nu}}$, with $\mathbb{H}$ acting on the left. Note that $X^{\mathbb{H}}(\nu) \simeq \mathbb{C}[W]$ as W-module, so $X^{\mathbb{H}}(\nu)$ contains the trivial W-type with multiplicity one.

If w_0 is the long Weyl group element, ν is dominant and $w_0 \cdot \nu = -\nu$, then $X^{\mathbb{H}}(\nu)$ has a unique irreducible quotient $L^{\mathbb{H}}(\nu)$.

Intertwining operators for affine graded Hecke algebras

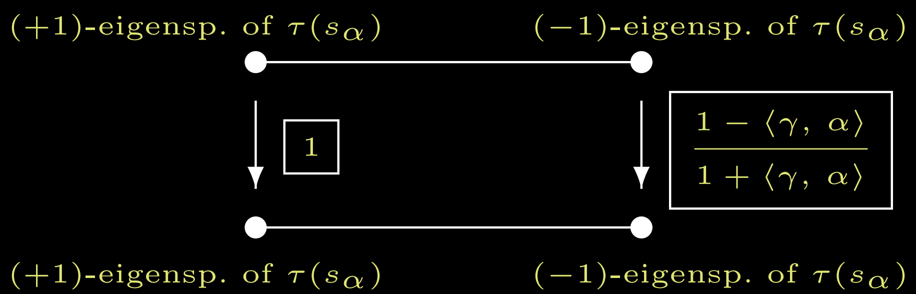
If ν is dominant and $w_0 \cdot \nu = -\nu$, the quotient $L^{\mathbb{H}}(\nu)$ is Hermitian. There is an operator

$$a(w_0, \nu) \colon X^{\mathbb{H}}(\nu) \to X^{\mathbb{H}}(-\nu)$$

which induces a non-degenerate invariant Herm. form on $X^{\mathbb{H}}(\nu)$.

Every W-type (τ, V_τ) inherits an operator $a_\tau(w_0, \nu)$ acting on V_τ^*. The Langlands quotient $X^{\mathbb{H}}(\nu)$ is unitary if and only if the operator $a_\tau(w_0, \nu)$ is positive semidefinite for every (relevant) W-type.

Note that $a_\tau(w_0, \nu) = \prod_{\alpha \text{ simple}} a_\tau(s_\alpha, \gamma)$, and $a_\tau(s_\alpha, \gamma)$ acts by:



Spherical Petite K-types for real split groups

Let μ be a spherical $\boldsymbol{K}$-type. The operator $\mathcal{A}_\mu(w, triv, \nu)$ acts on $(V_\mu^*)^M$. This space carries a representation ψ_μ of the Weyl group.

The spherical K-type μ is called "petite" if

$$\mathcal{A}_\mu(w, triv, \nu) = a_{\psi_\mu}(w, \nu).$$

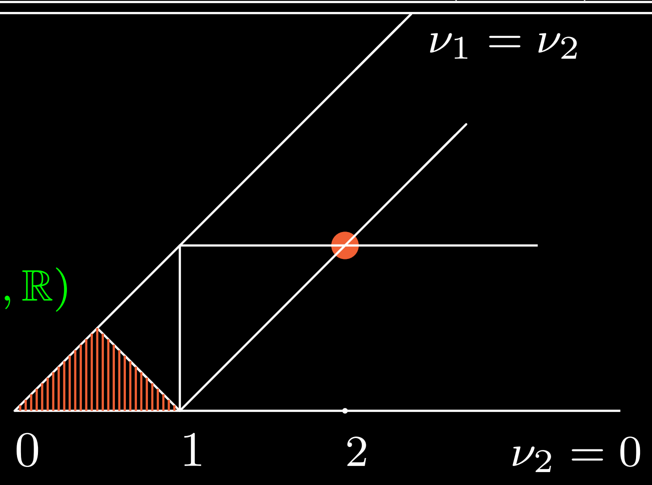
The latter is an operator for the affine graded Hecke algebra associated to G.

THEOREM. Spherical K-types of level at most 3 are petite.

An example: $G = Sp(4)$, $K = U(2)$, $W = W(C_2)$, $\delta = triv$

the petite K-type μ	the corresponding W-type ψ	the operator $a_\psi(\nu) = \mathcal{A}_\mu(w, triv, \nu)$
$(0,0)$	$(2)\times(0)$	1
$(1,-1)$	$(1,1)\times(0)$	$\frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)}\,\frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)}$
$(2,2)$	$(0)\times(2)$	$\frac{1-\nu_1}{1+\nu_1}\,\frac{1-\nu_2}{1+\nu_2}$
$(2,0)$	$(1)\times(1)$	*trace* $\quad 2\frac{1+\nu_1^2-\nu_1^3\nu_2-\nu_2^2+\nu_1\nu_2+\nu_1\nu_2^3}{(1+\nu_1)(1+\nu_2)[1+(\nu_1-\nu_2)][1+(\nu_1+\nu_2)]}$ *det* $\quad \frac{1-\nu_1}{1+\nu_1}\,\frac{1-\nu_2}{1+\nu_2}\,\frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)}\,\frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)}$

$L(triv,\nu)$ is unitary $\Rightarrow$
these 4 Hecke-algebra operators
are positive semidefinite
$\Rightarrow$ the spherical unitary dual of $Sp(4,\mathbb{R})$
is included in the set:



What spherical petite K-types do for us ...

unitarizability of **spherical** Langlands quotients for real split groups	$\overset{RELATE}{\Longleftrightarrow}$	unitarizability **spherical** Langlands quotients for affine graded Hecke algebras

Using petite K-types, Barbasch proves that the spherical unitary dual is always included in the the spherical unitary dual for an affine graded Hecke algebra. [This inclusion is an equality for classical groups.]

Some key facts:

1. (Barbasch, Barbasch-Ciubotaru) There is a small set of W-types (called "relevant") that detects unitarity for spherical Langlands quotients of Hecke algebras.

2. (Barbasch) For every relevant W-type τ there is a petite K-type μ s.t. the Hecke algebra operator a_τ matches the real operator $\mathcal{A}_\mu$.

Hence we always find an embedding of unitary duals:

$$\boxed{L^G(\nu) \text{ is unitary}} \quad =================> \quad \boxed{L^{\mathbb{H}}(\nu) \text{ is unitary}}$$

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$$

$$\boxed{\begin{matrix}\mathcal{A}_\mu \geq 0 \\ \forall \mu \in \widehat{K}\end{matrix}} \Rightarrow \boxed{\begin{matrix}\mathcal{A}_\mu \geq 0 \\ \forall \mu \text{ petite}\end{matrix}} \Rightarrow \boxed{\begin{matrix}a_\tau(\nu) \geq 0 \\ \forall \tau \text{ relevant}\end{matrix}} \Leftrightarrow \boxed{\begin{matrix}a_\tau \geq 0 \\ \forall \tau \in \widehat{W}\end{matrix}}$$

Non-spherical Petite K-types for real split groups

WISH LIST:

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- The operators $\{\mathcal{A}_\mu(\delta,\nu)\colon \mu \textit{ petite}\}$ should be "easy" to compute.
- The operators $\{\mathcal{A}_\mu(\delta,\nu)\colon \mu \textit{ petite}\}$ should rule out as many non-unitarity points as possible.
- The operators $\{\mathcal{A}_\mu(\delta,\nu)\colon \mu \textit{ petite}\}$ should relate the unitarizability of non-spherical Langlands subquotients for the real group with the unitarizability of certain Hecke algebras...

PROBLEM: How do we attach a Weyl group action to a K-type?

First guess: use the Weyl group of the good roots

Let δ be a non-spherical representation of M, and let μ be a K-type containing δ. The intertwining operator $\mathcal{A}_\mu(w, \delta, \nu)$ acts on the space $\mathrm{Hom}_M(\mu, \delta)$. We need some kind of Weyl group action on this space...

Let W^0_δ be the Weyl group of the good roots. Then W^0_δ acts *naturally* on $\mathrm{Hom}_M(\mu, \delta)$. [Call ψ^0_μ this W^0_δ-representation.]

A first attempt to define non-spherical petite K-types could be:

$$\mu \textit{ is petite } \Leftrightarrow A_\mu(w, \delta, \nu) = a_{\psi^0_\mu}(w, \nu).$$

This would not be a smart choice, because a parameter ν which is Hermitian for G may fail to be Hermitian for the affine graded Hecke algebra corresponding to W^0_δ.

Second guess: use the stabilizer of δ

Let W_δ be the stabilizer of the M-type δ. If we fix a fine K-type μ_δ containing δ, then we can let W_δ act on $\mathrm{Hom}_M(\mu, \delta)$ by $T \mapsto \mu_\delta(\sigma) T \mu(\sigma^{-1})$. [Call ψ_μ this W_δ-representation.]

Note that:

- ψ_μ depends on the choice of μ_δ, so ψ_μ *is not natural*.
- W_δ is the semidirect product of W_δ^0 by the R-group. The R-group is abelian if G is split, but may be non trivial. This forces us to *work with extended affine graded Hecke algebras*.

Nonetheless, the W_δ-type ψ_μ looks like the right object to consider. If we use W_δ, Hermitian parameters are preserved. Moreover, using W_δ, we carry along the action of the R-group and we can keep track of the (possible) reducibility of the Langlands quotient.

Extended affine graded Hecke algebras

Let $\mathbb{H} = \mathbb{C}W_\delta^0 \otimes A$ be the **affine graded Hecke algebra** associated to the root system of the good roots. Let $R = R_\delta$ be the R-group of δ. Then R is a finite abelian group acting on W_δ^0 and we can define:

$$\mathbb{H}' = \mathbb{C}[R] \ltimes \mathbb{H}.$$

For all $\nu \in \mathfrak{a}^*$, consider the principal series $\boldsymbol{X(\nu) :=} \mathbb{H}' \otimes_{\mathbb{A}'(\nu)} \mathbb{C}_{\boldsymbol{\nu}}$, with $\mathbb{H}$ acting on the left. Here $\mathbb{A}'(\nu) = \mathbb{C}[R(\nu)] \ltimes \mathbb{A}$ and R_ν is the centralizer of ν in R. Also note that the group

$$\mathbb{W}' = R \ltimes W$$

is isomorphic to W_δ. Suppose that $w = uw^0$ is a dominant element of W' such that $w\nu = -\nu$. For every $\psi' \in \widehat{W'}$, we have an operator $a_{\psi'}(uw^0, \nu)\colon \operatorname{Hom}_{W'}(\psi', X'(\nu)) \to \operatorname{Hom}_{W'}(\psi', X'(uw^0\nu))$. If $\psi^0 = \psi|_{W_\delta^0}$, then $a_{\psi'}(uw^0, \nu) = \psi'(u) a_{\psi^0}(w^0, \nu)$.

Non-spherical spherical petite K-types

- $\mathcal{A}_\mu(w,\delta,\nu)$ acts on the space $\mathrm{Hom}_M(\mu,\delta)$
  This space carries a representation $\boxed{\psi_\mu}$ of $W_\delta \leftarrow$ stabilizer of δ
- $\mathcal{A}_\mu(w, triv, \nu)$ only depends on the W-representation ψ_μ.
- $W_\delta = R \ltimes W_\delta^0 \leftarrow W_\delta^0 = W(good\,roots)$, $R \simeq R_\delta$.

  Define $\begin{cases} \psi_{\mu^0} = \text{restriction of } \psi_\mu \text{ to } W_\delta^0 \\ \psi_{\mu^R} = \text{restriction of } \psi_\mu \text{ to } W_\delta^R \end{cases}$
- Write $w = w^0 \cdot u$ with $w^0 \in W_\delta^0$ and $u \in R$.

Define:

$$\boxed{\mu \text{ petite}} \Leftrightarrow \boxed{\text{the real operator } \mathcal{A}_\mu(w,\delta,\nu) = \psi_\mu^R(u) a_{\psi_\mu^0}(w^0,\nu)}$$

The operator on right hand side can be interpreted as an operator for an extended Hecke algebra.

THEOREM. Non-spherical K-types of level less than or equal to 2 are petite for δ.

If μ is level 2, then $\mathcal{A}_\mu(w,\delta,\nu) = \psi_\mu^R(u)a_{\psi_\mu^0}(w^0,\nu)$.

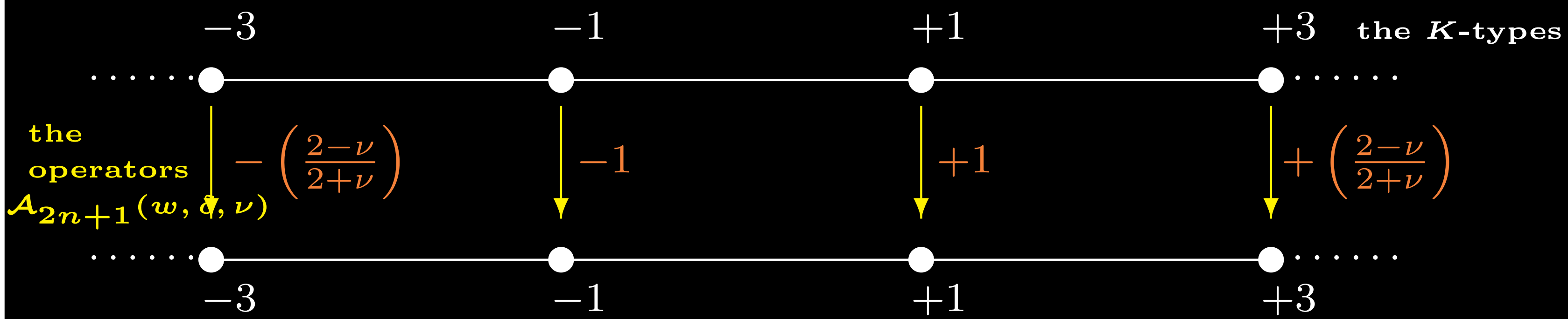
If μ is level 1 (fine), then $\mathcal{A}_\mu(uw^0,\delta,\nu) = \psi_\mu(u)$.

Non-spherical Langlands subquotients for $SL(2,\mathbb{R})$

$$G = SL(2,\mathbb{R}),\ K = SO(2,\mathbb{R}),\ \widehat{K} = \mathbb{Z},\ M = \mathbb{Z}_2,\ \delta = sign,\ \nu > 0$$

There is one operator $\mathcal{A}_{2n+1}(w,\delta,\nu)$ for every odd integer

The domain of $\mathcal{A}_{2n+1}(w,\delta,\nu)$ is 1-dimensional, so the operator $\mathcal{A}_{2n+1}(w,\delta,\nu)$ acts by a scalar:



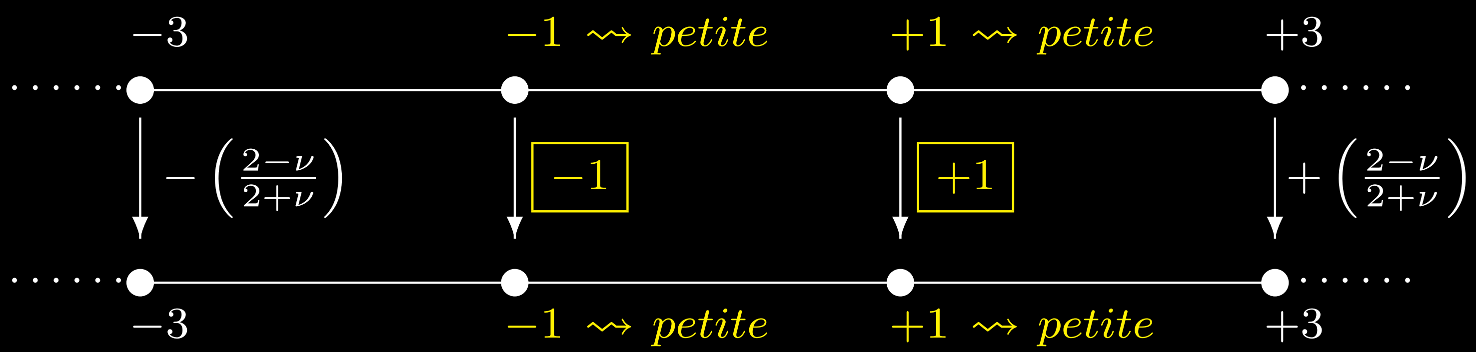
$L(\delta,\nu)$ is unitary $\Leftrightarrow \mathcal{A}_{2n+1}(w,\delta,\nu) \geq 0, \forall\, n$ $\Rightarrow$ *never unitary!*

Note that we could have used $\mu = +1$ and -1 alone!

Non-spherical petite K-types for $SL(2,\mathbb{R})$

We expect the K-types $\mu = +1$ and $\mu = -1$ to be ***petite***.

The corresponding operators $\mathcal{A}_\mu(w,\delta,\nu)$ are:



Note that:

- The K-types $\{\mu = \pm 1\}$ form a small set. ✓
- The operators $\{\mathcal{A}_\mu(w,\delta,\nu)\colon \mu = \pm 1\}$ are "easy". ✓
- The operators $\{\mathcal{A}_\mu(w,\delta,\nu)\colon \mu = \pm 1\}$ rule out **all** the non-unitarity points of $L(sign,\nu)$. ✓

So these K-types have all the desired properties (and more!).

What is special about the K-types ± 1?

$\forall \mu \in \widehat{K}$, $\mathcal{A}_\mu(w,\delta,\nu)$ is an operator on $\mathrm{Hom}_M(\mu,\delta)$. This space carries a representation ψ_μ of W_δ. If $\delta = sign$, then $W_\delta = R = W$ and $W_\delta^0 = \{1\}$. Note that $w = u \cdot 1 \in R$. On the fine K-types $\mu = \pm 1$, we have:

μ	ψ_μ	ψ_μ^0	ψ_μ^R	$\psi_\mu(u)$	
$+1$	$triv$	$triv$	$triv$	1	✓
-1	$sign$	$triv$	$sign$	-1	

An example: $G=Sp(4)$, $K=U(2)$, $W=W(C_2)$, $M=\mathbb{Z}_2^2$, $\underline{\nu=a>b\geq 0}$

If $\delta=(+,-)$, then $W_\delta=W(A_1)\times W(A_1)$, $W_\delta^0=W(A_1)=\langle s_{2e_1}\rangle$, $R=\mathbb{Z}_2=\langle s_{2e_2}\rangle$

petite K-type μ	ψ_μ: repr. of W_δ on $\mathrm{Hom}_M(\mu,\delta)$	ψ_μ^0: restriction of ψ_μ to W_δ^0	ψ_μ^R: restriction of ψ_μ to R
$(1,0)$	$triv\times triv$	$triv$	$triv$
$(0,-1)$	$triv\times sign$	$triv$	$sign$
$(2,1)$	$sign\times triv$	$sign$	$triv$
$(-1,-2)$	$sign\times sign$	$sign$	$sign$

Set $w_0=-I=s_{2e_1}s_{2e_2}=w^0u$ ($w^0=s_{2e_1}\in W_\delta^0$ and $u=s_{2e_2}\in R$.)

For μ petite: $A_\mu(w_0,\delta,\nu)=\psi_\mu^R(u)A_{\psi_\mu^0}(w^0,\nu)$

petite K-type μ	$\psi_\mu^R(u)$	$A_{\psi_\mu^0}(w^0,\nu)$	operator $A_\mu(w_0,\delta,\nu)$
$(1,0)$	$+1$	1	1
$(0,-1)$	-1	1	-1
$(2,1)$	$+1$	$\frac{1-a}{1+a}$	$\frac{1-a}{1+a}$
$(-1,-2)$	-1	$\frac{1-a}{1+a}$	$-\frac{1-a}{1+a}$

The 2 fine K-types have opposite sign. This is a problem iff they are *not* apart.

Let $\nu=(a,b)$ with $a > b \geq 0$. Then $R_\delta(\nu) = \begin{cases} \mathbb{Z}_2 & \text{if } b=0 \\ \{1\} & \text{if } b > 0. \end{cases}$

- If $b > 0$, the two fine K-types are contained in the same irreducible subquotient. The corresponding operators have opposite sign, so the quotient is not unitary.

- If $b = 0$, there are two irreducible subquotients: L_1 contains the K-types $(1,0)$ and $(2,1)$; L_2 contains the K-types $(0,-1)$ and $(-1,-2)$. The operators are

petite K-type μ	operator $A_\mu(w_0,\delta,\nu)$
$(1,0)$	1
$(2,1)$	$\frac{1-a}{1+a}$

petite K-type μ	operator $A_\mu(w_0,\delta,\nu)$
$(0,-1)$	-1
$(-1,-2)$	$-\frac{1-a}{1+a}$

We can use petite K-types to get necessary condition for unitarity, and deduce that neither L_1 nor L_2 are unitary if $a > 1$.
(If $0 \leq a \leq 1$, both L_1 and L_2 turn out to be unitary.)

An example: $G{=}Sp(4)$, $K{=}U(2)$, $W{=}W(C_2)$, $M{=}\mathbb{Z}_2^2$, $\underline{\nu{=}a > b \geq 0}$

If $\delta{=}(-,-)$, then $W_\delta{=}W$, $W_\delta^0{=}W(A_1) \times W(A_1){=}\langle s_{e_1-e_2}\rangle \times \langle s_{e_1+e_2}\rangle$, $R{=}\mathbb{Z}_2{=}\langle s_{2e_2}\rangle$.

Note that $w_0{=}{-I}{=}s_{e_1+e_2}s_{e_1-e_2}{=}w^0u$ ($w^0{=}{-I} \in W_\delta^0$ and $u{=}1 \in R$.)

For μ petite: $A_\mu(w_0,\delta,\nu) = \psi_\mu^R(u)A_{\psi_\mu^0}(w^0,\nu) = A_{\psi_\mu^0}(w^0,\nu)$.

petite μ	ψ_μ: repr. of W_δ on $\mathrm{Hom}_M(\mu,\delta)$	ψ_μ^0: restriction of ψ_μ to W_δ^0	$\psi_\mu^R(u)a_{\psi_\mu^0}(w^0,\nu) = a_{\psi_\mu^0}(w$
$(1,1)$	$(2)\times(0)$	$triv \times triv$	1
$(-1,-1)$	$(0)\times(2)$	$triv \times triv$	1
$(2,0)$	$(1,1)\times(0)$	$sign \times sign$	$\frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)}\,\frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)}$
$(0,-2)$	$(0)\times(1,1)$	$sign \times sign$	$\frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)}\,\frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)}$
$(1,-1)$	$(1)\times(1)$	$sign \times triv + triv \times sign$	$\left(\begin{matrix} \frac{1-(\nu_1-\nu_2)}{1+(\nu_1-\nu_2)} & 0 \\ 0 & \frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)} \end{matrix}\right.$

petite μ	ψ_μ: repr. of W_δ on $\mathrm{Hom}_M(\mu,\delta)$	ψ_μ^0: restriction of ψ_μ to W_δ^0
$(1,1)$	$(2)\times(0)$	$triv$
$(-1,-1)$	$(0)\times(2)$	$sign$
$(2,0)$	$(1,1)\times(0)$	$triv$
$(0,-2)$	$(0)\times(1,1)$	$sign$
$(1,-1)$	$(1)\times(1)$	$triv+sign$

If $\nu=(a,0)$, the Langlands quotient has two irreducible components.
$L(\delta,\nu)((1,1))$ contains 1 copy of (1,1), (2,0) and (1,-1).

$L(\delta,\nu)((-1,-1))$ contains 1 copy of (-1,-1), (0,-2) and (1,-1).
If $\nu=(a,b)$, with $b\neq 0$, the Langlands quotient is irreducible.

Embedding of unitary duals for Sp(2n)

Fix an M-type δ and a fine K-type μ_δ containing δ. Write $w = uw^0$, with $u \in R$ and $w^0 \in W_{\delta^0}$. For all μ petite,

$$\mathcal{A}^{\mathcal{G}}{}_{\mu}(w, \delta, \nu) = \psi^R_\mu(u)\mathcal{A}_{\psi^0_\mu}(w^0, \nu)$$

The second operator is an operator for an extended affine graded Hecke algebra $\mathbb{H}'(\delta)$ (associated to the stabilizer of δ).

If the R group is trivial, $\mathbb{H}'(\delta)$ is an honest affine graded Hecke algebra (associated to the system of good roots for δ).

If the R group is a $\mathbb{Z}_2$, we can regard $\mathbb{H}'(\delta)$ as a Hecke algebra with unequal parameters (the parameters being 0 or 1 depending on the length of the roots).

Notice that if $R_\delta(\nu) = \mathbb{Z}_2$, the matrices for the intertwining operators will have a block decomposition reflecting the multiplicity of the K-type in each of the two Langlands subquotients.

Like in the case of $Sp(4)$, one can try to use petite K-types to compare the set of unitary parameters for $X^G(\delta, \nu)$ with the set of quasi-spherical unitary parameters for $\mathbb{H}'(\delta)$.

It turns out that every relevant $W(\delta)$-type comes from petite K-type. So for $Sp(2n)$ one always obtain an embedding of unitary duals.