Unitary representations of real split groups

$SO(2, 3)_0$

$\nu_1 = \nu_2$

$\nu_2 = 0$

$Sp(4)$

$\nu_1 = \nu_2$

$\nu_2 = 0$

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Unitary Representations

$G$: a Lie group

- $S_n = \{\text{bijections on } \{1, 2, \ldots, n\}\}$ ← finite Lie group
- $S^1 = \{z \in \mathbb{C}: \|z\| = 1\}$ ← compact Lie group
- $SL(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}): \det A = 1\}$ ← noncompact Lie group

A unitary representation of $G$ on a Hilbert space $\mathcal{H}$ is a w. continuous action of $G$ on $\mathcal{H}$ by means of unitary operators

- $\mathcal{H} = \mathbb{C}$, $\pi(g)v = v$ ← trivial representation
- $\mathcal{H} = L^2(G, \mu)$, $\pi(g)f = f(\cdot g)$ ← right regular representation
Unitary Dual

\[ \hat{G}_u = \{ \text{equiv. classes of unitary irreducible repr.s of } G \} \]
PART 1

Motivation for the study the unitary dual

... from Fourier analysis to abstract harmonic analysis...
Classical Fourier Analysis: decompose a periodic function on \( \mathbb{R} \) in terms of trigonometric functions.

Abstract Harmonic Analysis: decompose \( L^2(G) \) in terms of unitary irreducible representations of \( G \).

\[
\begin{align*}
L^2(G) & \text{ for } G = \mathbb{R}/2\pi\mathbb{Z} \simeq S^1 \\
\uparrow & \\
\mathbf{f}(\theta) & = \sum_{n \in \mathbb{Z}} S_f(n) e^{i n \theta} \\
\uparrow & \\
\text{continuous function on } \mathbb{R} \text{ with period } 2\pi & \leftrightarrow \text{trigonometric functions} \\
\text{unitary irreducible repr.s of } G & \leftrightarrow \text{irreducible representations of } G
\end{align*}
\]

classical Fourier analysis \( \Rightarrow \) abstract harmonic analysis

trigonometric functions \( \Rightarrow \) unitary representations
Harmonic analysis on locally compact groups

$G$: abelian, compact, nilpotent, connected semisimple...

$$L^2(G) = \int_{\pi \in \hat{G}_u} \pi \, d\mu(\pi)$$

d$\mu$ is the *Plancharel measure* on $\hat{G}_u$.

$G$ is compact

$\hat{G}_u$ is a lattice

$d\mu(\pi) = \dim(\pi)$

$$L^2(G) = \bigoplus_{\pi \in \hat{G}_u} \dim(\pi) \pi$$

**Fourier Inversion Formula** . . . $G = S^1$, $f(\theta) = \sum_{n \in \mathbb{Z}} S_f(n) e^{in\theta}$
PART 2

Examples of unitary duals

- Finite groups
- Compact groups
- $SL(2, \mathbb{R})$
Unitary dual of FINITE groups

$G$: a finite group

- Every irreducible repr. of $G$ is finite-dim.l and unitary

$$\hat{G}_u=\{\text{finite-dim.l irreducible repr.s}\}/\text{equiv}$$

- $G$ has finitely many irreducible inequivalent repr.s

- The number of inequivalent irreducible repr.s of $G$ equals the number of conjugacy classes of $G$

Example: Let $G$ be “the monster”, a finite simple group containing almost $10^{54}$ elements. $G$ has 194 equivalence classes, so there are exactly 194 inequivalent irreducible unitary representations.
Unitary dual of the symmetric group $S_3$

$$(- - -) = \begin{array}{c}
\hline
\hline
\end{array}$$
trivial representation:
$$\mathcal{H} = \mathbb{C}, \; \rho(\sigma)v = v$$

$$(- -)(-) = \begin{array}{c}
\hline
\hline
\hline
\end{array}$$
permutation representation:
$$\mathcal{H} = \{v \in \mathbb{C}^3 : \sum_{i=1}^{3} v_i = 0\}, \; \rho(\sigma)(v) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$$

$$(-)(-)(-) = \begin{array}{c}
\hline
\hline
\hline
\end{array}$$
sign representation:
$$\mathcal{H} = \mathbb{C}, \; \rho(\sigma)v = sgn(\sigma)v$$
Unitary dual of COMPACT groups

\( G \): a \((\text{non-finite})\) compact group, with maximal torus \( T \)

- Every irreducible repr. of \( G \) is finite-dim.l and unitary

\( \hat{G}_u = \{\text{finite-dim.l irreducible repr.s}\}/\text{equiv} \)

- \( G \) has \underline{infinitely many}\ irreducible inequivalent repr.s

\[ \begin{align*}
  G = S^1 &\Rightarrow \forall n \in \mathbb{Z}, \pi_n : S^1 \to \mathbb{C}^*, \ e^{i\theta} \mapsto e^{in\theta} 
\end{align*} \]

- \( \hat{G}_u \) is parameterized by the lattice of dominant weights:

\[ \mathcal{C} = \{\lambda \in t^* : \lambda = \text{differential of a character of } T, \text{ and } \lambda \text{ is dominant}\} \]

There is a bijection \( \hat{G}_u \to \mathcal{C}, \pi \mapsto \lambda_\pi = \text{highest weight of } \pi. \)
What about the non-compact group $SL(2, \mathbb{R})$?

$G = SL(2, \mathbb{R}) = \{2 \times 2 \text{ real matrices with determinant 1}\}$

$SL(2, \mathbb{R})$ has only one finite-dimensional unitary irreducible representation: the trivial representation!
Unitary dual of $SL(2, \mathbb{R})$

Bargmann, 1947

Non-spherical principal series

$spherical principal series$

$P_{i\nu}^-$
$\nu > 0$

$P_{i\nu}^+$
$\nu \geq 0$

holomorphic discrete series

anti-holomorphic discrete series

$D_1^+$
$D_2^+$
$D_3^+$
$D_n^+$

limits of discrete series

complementary series

Only the trivial representation is finite-dimensional!
Compact/Non-compact groups

COMPACT (or FINITE) groups: Every irreducible unitary representation is finite-dimensional. Moreover, every finite-dimensional representation is unitarizable.

- Start from any inner product $(\cdot, \cdot)$ on $\mathcal{H}$
- Construct an invariant inner product $\langle \cdot, \cdot \rangle$ by averaging:
  \[
  \langle v, w \rangle \equiv \frac{1}{\#G} \sum_{x \in G} (\pi(x)v, \pi(x)w) \quad \forall v, w \in \mathcal{H}.
  \]
  For compact groups, replace $\sum_{x \in G}$ by $\int_G \cdot d\mu$.

NON-COMPACT linear semisimple groups: Every nontrivial irreducible unitary repr. is infinite-dim.l. Moreover, not every infinite-dimensional representation is unitary.

$\Rightarrow$ finding the unitary dual of non-compact groups is much harder
Unitary dual of real reductive groups

$G$: real reductive group, e.g. $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $Sp(n)$
or any closed subgroup of $GL(n, \mathbb{C})$ stable under $A \mapsto (A^t)^{-1}$

$\hat{G}_u = ?$

A complete answer is only known for

- $SL(2, \mathbb{R}) \leftarrow$ Bargmann, 1947
- $GL(n, \mathbb{R}) \leftarrow$ Vogan, 1986
- complex classical groups $\leftarrow$ Barbasch, 1989
- $G_2 \leftarrow$ Vogan, 1994
PART 3

Progress in the classification of the unitary dual
The greatest heros

- **Harish-Chandra [1952]:** Algebraic reformulation of the problem of finding the unitary dual

\[
\hat{G}_u = \left\{ \begin{array}{l}
\text{unitary irred.} \\
\text{repr.s of } G
\end{array} \right\} \quad = \left\{ \begin{array}{l}
\text{unitary irred.} \\
(g, K)\text{-modules}
\end{array} \right\}
\]

- **Langlands [1973]:** Classification of irreducible \((g, K)\)-modules

- **Knapp and Zuckerman [1976]:** Classification of Hermitian irreducible \((g, K)\)-modules
Sketch of the history

\( \hat{G}_{\text{unitary}} \)

1952 \( \parallel \) H.C.

unitary irr.
\((g, K)\)-mod.s

\( \subseteq \)

unitary
L. quotients

1976 \( \uparrow \) \( K. \mathbb{Z} \)

To get \( \hat{G}_u \), we need to find which Langlands quotients are unitary.
(minimal) Langlands Quotients with real inf. character

parameters
\[
\begin{align*}
P &= MAN & \text{minimal parabolic subgroup of } G \\
(\delta, V^\delta) & \quad \text{irreducible representation of } M \\
\nu : \mathfrak{a} & \to \mathbb{R} & \text{strictly dominant linear functional}
\end{align*}
\]

principal series
\[
I_P(\delta, \nu) = \text{Ind}_{P=MAN}^G (\delta \otimes \nu \otimes \text{triv})
\]

\[G \text{ acts by left translation on:}
\{ F : G \to V^\delta : F|_K \in L^2, F(x\text{man}) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(x), \ \forall \text{man} \in P \}\]

Langlands quotient
\[
J(\delta, \nu) : 1! \text{ irred. quotient of } I_P(\delta, \nu)
\]

intertwining operator
\[
A(\delta, \nu) : I_P(\delta, \nu) \to I_{\bar{P}}(\delta, \nu), \ F \mapsto \int_{\bar{N}} F(x\bar{n}) \ d\bar{n}
\]
\[
J(\delta, \nu) \equiv \frac{I_P(\delta, \nu)}{\ker A(\delta, \nu)}
\]

The Hermitian form on \(J(\delta, \nu)\) is induced by the operator \(A(\delta, \nu)\)
A more informal definition (for real split groups)

**Langlands Quotients** $J(\delta, \nu) = \text{Candidates for Unitarity}$

↑

**irreducible repr.s of** $G$ **parameterized by:**

- $\boxed{P} = MAN$: a (fixed) minimal parabolic subgroup of $G$
- $\boxed{\delta}$: an irreducible representation of the finite group $M$
- $\boxed{\nu}$: an element of a cone, in a vector space of dim.$=\text{rank}(G)$

If $J(\delta, \nu)$ is Hermitian, the form is induced by an operator $A(\delta, \nu)$.

**Unitary dual Problem:** finding all $\delta$ and $\nu$ s.t.

$$J(\delta, \nu) \text{ is unitary} \iff A(\delta, \nu) \text{ is pos. semidefinite}$$

**Hard Problem:** *the set of unitary parameters is very small.*
Spherical unitary duals of $SL(2, \mathbb{R})$, $SO(2,3)_0$ and $Sp(4)$

Fix $P = MAN$: minimal parabolic subgroup, $\delta$: trivial repr. of $M$. The only parameter is $\nu$; $\nu$ varies in a cone inside a vector space of $\text{dim.} = \text{rank}(G)$. Consider the Langlands quotient $J(\nu) \equiv J(\text{triv}, \nu)$. The values of $\nu$ s.t. $J(\nu)$ is unitary are painted in red:

$SL(2, \mathbb{R})$ rank 1

$SO(2,3)_0$ rank 2

$Sp(4)$ rank 2
PART 4

Find the Spherical Unitary Dual

i.e. discuss the unitarity of a spherical Langlands quotient $J(\nu)$
**Spherical Unitary Dual of split groups**

- **The setting:**
  
  \( G \): a split connected real reductive group  
  
  \( K \): maximal compact subgroup  
  
  \( (G = SL(n, \mathbb{R}), K = SO(n)) \)

- **The problem:**

  Find the spherical unitary dual of \( G \). A representation of \( G \) is called **spherical** if it contains the trivial representation of \( K \).

- **Candidates:** spherical Langlands quotients \( J(\nu) \equiv J(\text{triv}, \nu) \)

  \[ \Rightarrow \text{Equivalent problem:} \quad \text{Find all} \ \nu \ \text{such that} \ J(\nu) \ \text{is unitary} \]

- **Status quo:**

  By work of Knapp and Zuckermann, we know which \( J(\nu) \)'s are Hermitian, i.e. have an invariant Hermitian form.
  
  If \( J(\nu) \) is Hermitian, the form is induced by the operator \( A(\nu) \).
  
  So \( J(\nu) \) is unitary iff \( A(\nu) \) is positive semidefinite.
Studying the signature of the operator $A(\nu)$

- The operator $A(\nu)$ acts on the spherical principal series, which is an infinite-dimensional vector space.

- $A(\nu)$ preserves the isotypic component of the various $K$-types $\mu \in \hat{K}$ that appear in the spherical principal series (“spherical $K$-types”).

- There are infinitely many spherical $K$-types, but each appears with finite multiplicity.

- Restricting the operator $A(\nu)$ to the isotypic component of $\mu$, we get an operator $A_\mu(\nu)$ for each spherical $K$-type $\mu$.

$J(\nu)$ is unitary iff $A_\mu(\nu)$ is positive semidefinite for all $\mu$. 

The example of \( SL(2, \mathbb{R}) \)

\[ G = SL(2, \mathbb{R}), \quad K = SO(2, \mathbb{R}), \quad \hat{K} = \mathbb{Z}, \quad \hat{K}_{spherical} = 2\mathbb{Z} \]

There is one operator \( A_{2n}(\nu) \) for every even integer

Each operator \( A_{2n}(\nu) \) acts by a scalar:

\[
\begin{align*}
0 & \quad \pm 2 & \quad \pm 4 & \quad \pm 6 & \quad \text{the } K\text{-types } 2n \\
& \quad \downarrow & \quad \downarrow & \quad \downarrow & \\
1 & \quad \frac{1-\nu}{1+\nu} & \quad \frac{(1-\nu)(3-\nu)}{(1+\nu)(3+\nu)} & \quad \frac{(1-\nu)(3-\nu)(5-\nu)}{(1+\nu)(3+\nu)(5+\nu)}
\end{align*}
\]

The representation \( J(\nu) \) is unitary iff every \( A_{2n}(\nu) \) is \( \geq 0 \)

\[
0 \leq \nu \leq 1
\]
Other real split reductive groups

- There are infinitely-many spherical $K$-types $\mu$
- For each $\mu$, there is an operator $A_{\mu}(\nu)$
- The formula for $A_{\mu}(\nu)$ becomes very complicated if $\mu$ is “big”
- To obtain necessary and sufficient conditions for unitarity, one needs to study the signature of the operator $A_{\mu}(\nu)$ for all $\mu$

*Vogan, Barbasch:* Only look for necessary conditions for unitarity

⇓

Isolate finitely many $K$-types $\mu$ (called “petite”) s.t. the operator $A_{\mu}(\nu)$ is easy. Only compute the signature of $A_{\mu}(\nu)$ for $\mu$ petite.
Spherical Petite $K$-types for $SL(2, \mathbb{R})$

$G = SL(2, \mathbb{R}), \hat{K}_{spherical} = 2\mathbb{Z}$. Spherical petite $K$-types: $n = 0, \pm 2$

- **Necessary and Sufficient conditions for unitarity:**
  
  $A_{2n}(\nu)$ is pos. semidefinite for every $K$-type $2n$

- **Necessary conditions for unitarity:**
  
  $A_{2n}(\nu)$ is pos. semidefinite for the petite $K$-type $2n = 0, \pm 2$
**Definition** [Barbasch, Vogan] For every root $\alpha$, there is a subgroup $K_\alpha \simeq SO(2)$. A spherical $K$-type $\mu$ is called **petite** if the restriction of $\mu$ to $K_\alpha$ only contains the $SO(2)$-types 0 and $\pm 2$.

If $\mu$ is petite, the intertwining operator $A_\mu(\nu)$ is "easy" to compute.

**Easy**: $A_\mu(\nu)$ behaves exactly like an operator for a $p$-adic group.
The operator $A_\mu(\nu)$ on a petite $K$-type $\mu$

• $A_\mu(\nu)$ acts on the space $\text{Hom}_M(\mu, \mathbb{C}) = (V_\mu^*)^M$.

• This space carries a representation $\psi_\mu$ of the Weyl group $W$.

• $A_\mu(\nu)$ only depends on the $W$-representation $\psi_\mu$.

Indeed, we can compute $A_\mu(\nu)$ by means of Weyl group calculations:

\[
A_\mu(\nu) = \prod_{\alpha \text{ simple}} A_\mu(s_\alpha, \gamma)
\]

For $p$-adic groups, there is an operator $A_{\psi}(\nu)$ for each $W$-type $\psi$.

$\mu$ petite $\Rightarrow$ the real operator $A_\mu(\nu) = \text{the } p\text{-adic operator } A_{\psi_\mu}(\nu)$
Comparing spherical unitary duals (real ↔ \(p\)-adic)

- The unitarity of a Langlands quotient \(J(\nu)\) for a split group \(G\) depends on the signature of some intertwining operators.
- For real groups there is an operator \(A_{\mu}(\nu)\) for every irreducible representation \(\mu\) of the maximal compact subgroup \(K\).
- For \(p\)-adic groups there is an operator \(A_{\psi}(\nu)\) for every irreducible representation \(\psi\) of the Weyl group \(W\).

It is enough to consider “relevant” \(W\)-types, because relevant \(W\)-types detect unitarity.

- \([Barbasch]\) Every relevant \(W\)-type is matched with a petite \(K\)-type s.t. the corresponding intertwining operators coincide.
- This matching implies an inclusion of spherical unitary duals.
(⋆) For each relevant $W$-type $\psi$, there is a petite $K$-type $\mu$ s.t. the p-adic operator on $\psi = $ the real operator on $\mu$.

(⋆) Relevant $W$-types detect unitarity.
An embedding of spherical unitary duals for split groups

\[ \text{spherical unitary dual of } G(\mathbb{R}) \subseteq \text{spherical unitary dual of } G(\mathbb{Q}_p) \]

[Barbasch]: this inclusion is an equality for classical groups

The spherical unitary dual of a split \( p \)-adic group is known. Then

- for classical real split groups, one obtains the full spherical unitary dual

- for non-classical real split groups, one obtains strong necessary conditions for the unitarity of a spherical Langlands quotient.
PART 5

Find the Non-Spherical Unitary Dual

i.e. discuss the unitarity of a Langlands quotient $J(\delta, \nu)$, $\delta \neq \text{triv}$
Like in the spherical case, we need to understand which Hermitian Langlands quotients \( J(\delta, \nu) \) are unitary.

To find necessary and sufficient conditions for unitarity, one needs to compute the signature of infinitely many operators.

There is an operator \( A_{\mu}(\delta, \nu) \) for every \( K \)-type \( \mu \) containing \( \delta \).

If \( \mu \) is “big”, computing \( A_{\mu}(\delta, \nu) \) is extremely hard.

Instead, we (only) look for necessary conditions...
Necessary conditions for unitarity

**Spherical case - Vogan, Barbasch:**

Define “spherical petite $K$-types”, and use them to compare the spherical unitary dual of a real split group $G$ with the spherical unitary dual of the corresponding p-adic group.

**Non-spherical case - P., Barbasch:**

Define “non-spherical petite $K$-types”, and use them to compare the non-spherical unitary dual of a real split group $G$ with the spherical unitary dual of a (different) p-adic group.
Non-spherical unitary dual

For each $\delta$, we construct a $p$-adic group $G(\delta)_{Q_p}$. Then we define non-spherical petite K-types (for $\delta$) and we use them to compare

the non-spherical unitary dual of $G_{R}$ induced by $\delta$

with

the spherical unitary dual of $G(\delta)_{Q_p}$

\[\uparrow\]

$candidates : J(\delta, \nu)_{G_{R}}$

\[J(\delta, \nu)_{G_{R}} \text{ unitary } \iff \quad A_{\mu}(\delta, \nu) \geq 0, \; \forall \mu \in \hat{K}\]

\[\uparrow\]

$candidates : J(\nu_{0})_{G(\delta)_{Q_p}}$

\[J(\nu_{0})_{G(\delta)_{Q_p}} \text{ unitary } \iff \quad A_{\psi}(\nu) \geq 0, \; \forall \psi \in (W_{0})_{rel}\]
One doesn’t always get an embedding of unitary duals

\[ J(\delta, \nu) \text{ is unit. for } G_{\mathbb{R}} \quad \iff \quad J(\text{triv}, \nu_0) \text{ is unit. for } G(\delta)_{\mathbb{Q}_p} \]

| \( A_\mu(\delta, \nu) \geq 0 \) | \( A_\psi(\nu_0) \geq 0 \)  
|-----------------------|-----------------------|
| \( \forall \mu \in \hat{K} \) | \( \forall \psi \in \hat{W}_0 \)  

\[ A_\mu(\delta, \nu) \geq 0 \quad \forall \mu \text{ petite} \]

\[ A_\psi(\nu_0) \geq 0 \quad \forall \psi \text{ relev} \]

if you have enough petite \( K \)-types to match all the relevant \( W_0 \)-types
The linear split group \( F_4 \)

- \( G = F_4 \)
- \( K = [Sp(1) \times Sp(3)]/\{\pm I\} \)

\( K \)-types = irreducible repr.s of \( K \), classified by highest weight:
\[ \mu = (a_1|a_2, a_3, a_4), \text{ with } a_1 \geq 0, a_2 \geq a_3 \geq a_4 \geq 0, \sum a_i \equiv 0 \]

**Minimal Principal Series:** \( I(\delta, \nu) \)

- \( P = MAN \) = a minimal parabolic subgroup
- \( M \): a finite abelian group of order 16
- \( A \): vector group \( (\dim \text{Lie}(A) = 4) \)
- \( \delta \): irreducible representation of \( M \)
- \( \nu \): dominant linear functional on \( \text{Lie}(A) \)

**Problem:** discuss the unitarity of the Langlands quotients \( J(\delta, \nu) \)
The Weyl group $W$ acts on $\hat{M}$. Let $W(\delta)$ be the stabilizer of $\delta$.

- $W(\delta)$ only depends on the $W$-orbit of $\delta$
- $W(\delta)$ is the Weyl group of a root system $\Delta_0(\delta)$.

Let $G(\delta)$ be the corresponding split group.

<table>
<thead>
<tr>
<th>representative for the $W$-orbit of $\delta$</th>
<th>root system $\Delta_0(\delta)$</th>
<th>corresponding split group $G(\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_1$</td>
<td>$F_4$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$\delta_3$</td>
<td>$C_4$</td>
<td>$Sp(4)$</td>
</tr>
<tr>
<td>$\delta_{12}$</td>
<td>$B_3A_1$</td>
<td>$SO(4,3)_o \times SL(2)$</td>
</tr>
</tbody>
</table>

Using petite $K$-types, we relate the unitarity of a (possibly non-spherical) Langlands quotient of $G$ induced from $\delta$ to the unitarity of a spherical Langlands quotient of $G(\delta)$.
Examples for the linear split group $F_4$

- $\delta = \delta_{12}; \ G(\delta) = SO(4,3)_0 \times SL(2)$

- Every relevant $W$-type for $G(\delta)$ can be matched with a petite $K$-type for $F_4$. Hence there is an inclusion of unitary duals:

  \[
  \begin{array}{c}
  \text{unitary parameters} \\
  \text{for } (\delta_{12}, F_4)
  \end{array}
  \subset
  \begin{array}{c}
  \text{unitary parameters} \\
  \text{for } (\text{triv}, SO(4,3)_0 \times SL(2))
  \end{array}
  \]

- $\delta = \delta_3; \ G(\delta) = Sp(4)$

- We can match every relevant $W$-type for $G(\delta)$ except $1 \times 3$. Hence we obtain a weaker inclusion:

  \[
  \begin{array}{c}
  \text{unitary parameters} \\
  \text{for } (\delta_3, F_4)
  \end{array}
  \subset
  \begin{array}{c}
  \text{unitary parameters} \\
  \text{for } (\text{triv}, Sp(4))
  \end{array}
  \cup
  \begin{array}{c}
  \text{non-unitarity region} \\
  \text{for } (\text{triv}, Sp(4)) \text{ ruled out by } 1 \times 3
  \end{array}
  \]
Conclusions

A Hermitian representation is unitary if and only if the invariant Hermitian form is positive definite.

Using petite $K$-types, we compare invariant forms on Hermitian representations for real and p-adic groups.

This comparison leads to interesting relations between the unitary duals of the two groups.

For example, it implies that the spherical unitary dual of a real split group is always contained in the spherical unitary dual of the corresponding p-adic group.

In the non-spherical case, you still get very interesting inclusions.