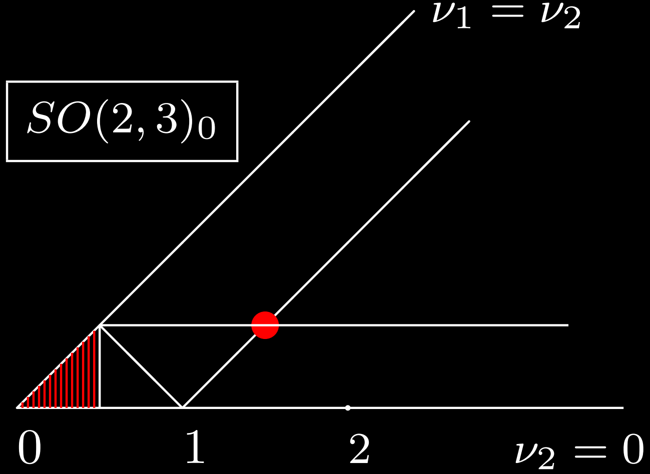
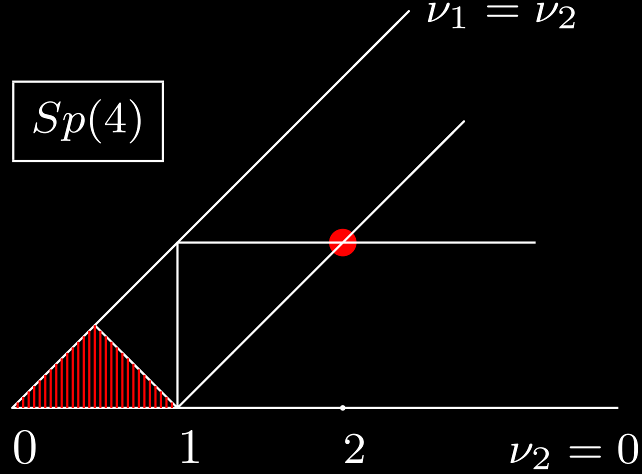
Unitary representations of real split groups





Alessandra Pantano, UCI

December 2007

Unitary Representations

G : a Lie group

- $\mathcal{S}_n = \{\text{bijections on } \{1, 2, \ldots, n\}\}$ ← *finite Lie group*
- $\mathcal{S}^1 = \{z \in \mathbb{C} \colon \|z\| = 1\}$ ← *compact Lie group*
- $SL(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}) \colon \det A = 1\}$ ← *noncompact Lie group*

A **unitary representation** of G on a Hilbert space $\mathcal{H}$ is a w. continuous action of G on $\mathcal{H}$ by means of unitary operators

- $\mathcal{H} = \mathbb{C}$, $\pi(g)v = v$ ← *trivial representation*
- $\mathcal{H} = L^2(G, \mu)$, $\pi(g)f = f(\cdot\, g)$ ← *right regular representation*

Unitary Dual

$$\boxed{\widehat{G}_u = \{\text{equiv. classes of unitary irreducible repr.s of } G\}}$$

?

PART 1

Motivation for the study the unitary dual

... from Fourier analysis to abstract harmonic analysis...

Classical Fourier Analysis — decompose a periodic function on $\mathbb{R}$ in terms of trigonometric functions

Abstract Harmonic Analysis — decompose $L^2(G)$ in terms of unitary irreducible repr.s of G

$$\underset{\substack{L^2(G)\ \text{for}\\ G=\mathbb{R}/2\pi\mathbb{Z}\simeq S^1}}{} \longleftarrow \underset{\substack{\uparrow\\ \text{continuous function on } \mathbb{R} \text{ with period } 2\pi}}{\boxed{f(\theta)}} = \sum_{n\in\mathbb{Z}} S_f(n)\ \underset{\substack{\uparrow\\ \text{trigonometric functions}}}{\boxed{e^{in\theta}}} \hookrightarrow \text{unitary irreducible repr.s of } G$$

classical Fourier analysis		abstract harmonic analysis
trigonometric functions	$\Rightarrow$	**unitary representations**

Harmonic analysis on locally compact groups

G: *abelian, compact, nilpotent, connected semisimple...*

$$\boldsymbol{L^2(G) = \int^{\oplus}_{\pi \in \widehat{G}_u} \pi \, \mathrm{d}\mu(\pi)}$$

$\mathrm{d}\mu$ is the *Plancharel measure* on $\widehat{G}_u$.

G is compact

$\widehat{G}_u$ is a lattice

$\mathrm{d}\mu(\pi) = \dim(\pi)$

$L^2(G) = \bigoplus_{\pi \in \widehat{G}_u} \dim(\pi)\pi$

Fourier Inversion Formula ... $G = S^1$, $f(\theta) = \sum_{n \in \mathbb{Z}} S_f(n)\, e^{in\theta}$

PART 2

Examples of **unitary duals**

- Finite groups
- Compact groups
- $SL(2, \mathbb{R})$

Unitary dual of FINITE groups

G: a finite group

- **Every irreducible repr. of G is finite-dim.l and unitary**

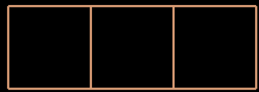
$$\boxed{\widehat{G}_u = \{\text{finite-dim.l irreducible repr.s}\}/\text{equiv}}$$

- **G has finitely many irreducible inequivalent repr.s**

- **The number of inequivalent irreducible repr.s of G equals the number of conjugacy classes of G**

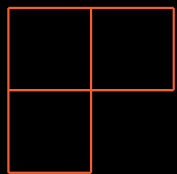
Example: Let G be "the monster", a finite simple group containing almost 10^{54} elements. G has 194 equivalence classes, so there are exactly 194 inequivalent irreducible unitary representations.

Unitary dual of the symmetric group $\mathcal{S}_3$

$(-\ -\ -) \quad =$  $\quad \leftrightsquigarrow$

trivial representation:

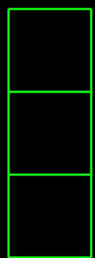
$\mathcal{H} = \mathbb{C},\ \rho(\sigma)v = v$

$(-\ -)(-) \quad =$  $\quad \leftrightsquigarrow$

permutation representation:

$$\mathcal{H} = \{\underline{v} \in \mathbb{C}^3 \colon \sum_{i=1}^{3} v_i = 0\}$$

$$\rho(\sigma)(\underline{v}) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$$

$(-)(-)(-) \quad =$  $\quad \leftrightsquigarrow$

sign representation:

$\mathcal{H} = \mathbb{C},\ \rho(\sigma)v = sgn(\sigma)v$

Unitary dual of COMPACT groups

G: a (non-finite) compact group, with maximal torus T

- **Every irreducible repr. of G is finite-dim.l and unitary**

$$\boxed{\widehat{G}_u = \{\text{finite-dim.l irreducible repr.s}\}/\text{equiv}}$$

- **G has infinitely many irreducible inequivalent repr.s**

e.g. $G = \mathcal{S}^1 \Rightarrow \forall\, n \in \mathbb{Z},\ \pi_n : \mathcal{S}^1 \to \mathbb{C}^\star,\ e^{i\theta} \mapsto e^{in\theta}$

- **$\widehat{G}_u$ is parameterized by the lattice of dominant weights:**

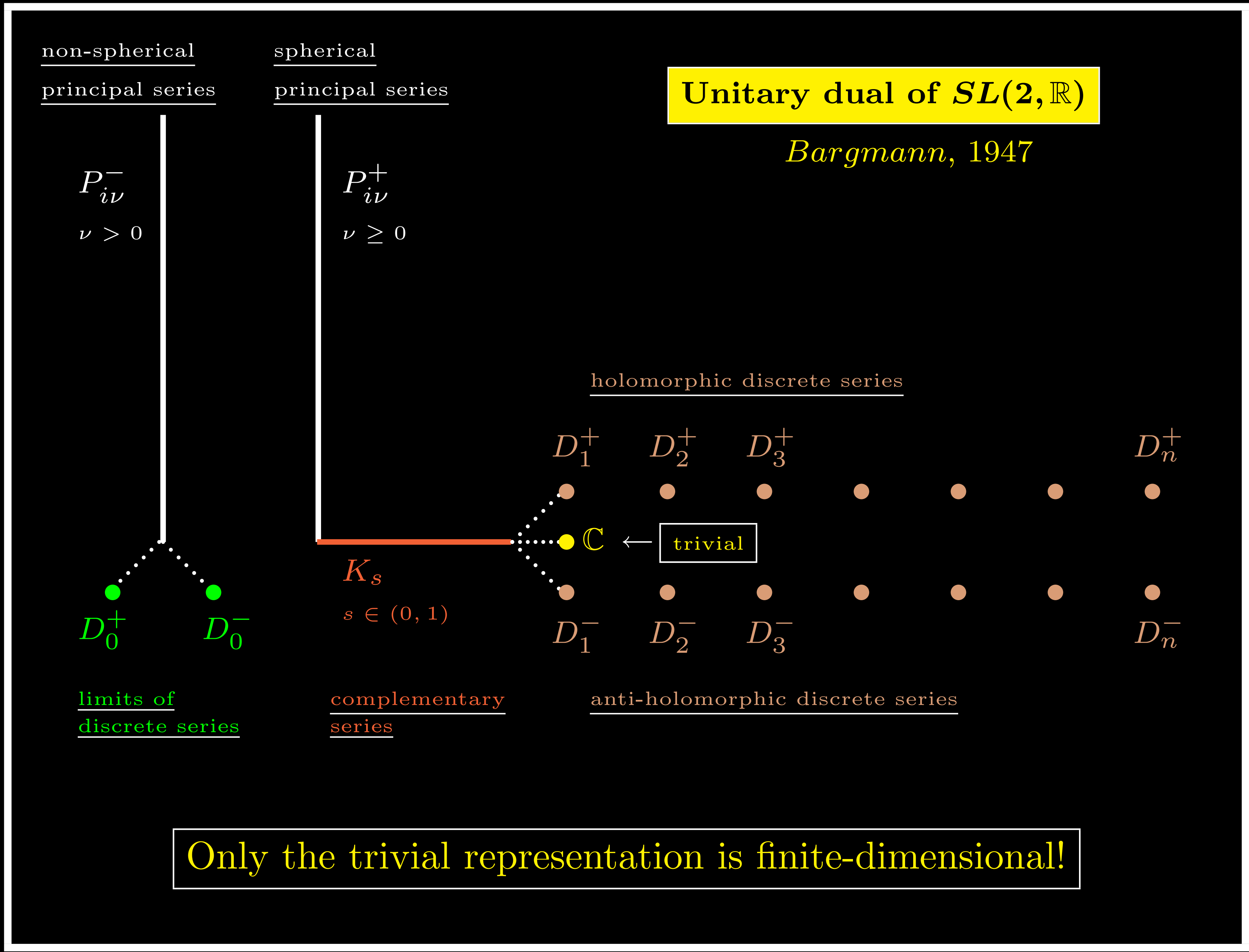
$$\boxed{\mathcal{C} = \{\lambda \in \mathfrak{t}^* : \lambda = \text{differential of a character of } T, \text{ and } \lambda \text{ is dominant}\}}$$

There is a bijection $\widehat{G}_u \to \mathcal{C}$, $\pi \mapsto \lambda_\pi =$ highest weight of π.

What about the non-compact group $SL(2,\mathbb{R})$?

$$G = SL(2,\mathbb{R}) = \{2 \times 2 \text{ real matrices with determinant } 1\}$$

$SL(2,\mathbb{R})$ has only one finite-dimensional unitary irreducible representation: the trivial representation!


Unitary dual of $SL(2, \mathbb{R})$
Bargmann, 1947
non-spherical
principal series
spherical
principal series
$P^-_{i\nu}$
$\nu > 0$
$P^+_{i\nu}$
$\nu \geq 0$
holomorphic discrete series
D^+_1 D^+_2 D^+_3 D^+_n
$\mathbb{C}$ ⟵ trivial
K_s
$s \in (0, 1)$
D^+_0 D^-_0
D^-_1 D^-_2 D^-_3 D^-_n
limits of
discrete series
complementary
series
anti-holomorphic discrete series
Only the trivial representation is finite-dimensional!

Compact/Non-compact groups

COMPACT (or FINITE) groups: Every irreducible unitary representation is finite-dimensional.

Moreover, every finite-dimensional representation is unitarizable.

- *Start from any inner product* $(\cdot,\cdot)$ *on* $\mathcal{H}$
- *Construct an invariant inner product* $\langle\cdot,\cdot\rangle$ *by averaging:*

$$\langle v, w\rangle \equiv \frac{1}{\#G} \sum_{x\in G} (\pi(x)v, \pi(x)w) \quad \forall\, v, w \in \mathcal{H}.$$

For compact groups, replace $\sum_{x\in G}$ by $\int_G \cdot\, \mathrm{d}\mu$.

NON-COMPACT linear semisimple groups: Every *nontrivial* irreducible unitary repr. is infinite-dim.l.

Moreover, not every infinite-dimensional representation is unitary.

$\Rightarrow$ finding the unitary dual of non-compact groups is much harder

Unitary dual of real reductive groups

G: real reductive group, e.g. $SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $Sp(n)$ or any closed subgroup of $GL(n,\mathbb{C})$ stable under $A \mapsto \overline{(A^t)^{-1}}$

$$\boxed{\widehat{G}_u = ?}$$

A complete answer is only known for

- $SL(2,\mathbb{R}) \leftarrow$ Bargmann, 1947
- $GL(n,\mathbb{R}) \leftarrow$ Vogan, 1986
- complex classical groups $\leftarrow$ Barbasch, 1989
- $G_2 \leftarrow$ Vogan, 1994

PART 3

Progress in the classification of the unitary dual

The greatest heros

- Harish-Chandra *[1952]: Algebraic reformulation of the problem of finding the unitary dual*

$$\widehat{G}_u = \left\{ \begin{array}{c} \text{unitary irred.} \\ \text{repr.s of } G \end{array} \right\}_{unit.\ equiv} = \left\{ \begin{array}{c} \text{unitary irred.} \\ (\mathfrak{g}, K)\text{-modules} \end{array} \right\}_{equiv}$$

- Langlands *[1973]: Classification of irreducible $(\mathfrak{g}, K)$-modules*

- Knapp and Zuckerman *[1976]: Classification of Hermitian irreducible $(\mathfrak{g}, K)$-modules*

Sketch of the history

$\widehat{G}_{unitary}$

1952 $\|$ *H.C.*

unitary irr. $(\mathfrak{g}, K)$*-mod.s*	$\subseteq$	*Hermitian irr.* $(\mathfrak{g}, K)$*-mod.s*	$\subseteq$	*irreducible* $(\mathfrak{g}, K)$*-mod.s*
$\|$		$\|$		*L.* $\|$ *1973*
unitary L. quotients		*Hermitian L. quotients*		*Langlands quotients*
$\uparrow$		*1976* $\uparrow$ *K.& Z.*		$\uparrow$
?		$\checkmark$		$\checkmark$

To get $\widehat{G}_u$, we need to find which Langlands quotients are unitary.

(minimal) Langlands Quotients with real inf. character

- **parameters** $\begin{cases} P = MAN & \text{minimal parabolic subgroup of } G \\ (\delta,\, V^\delta) & \text{irreducible representation of } M \\ \nu\colon \mathfrak{a} \longrightarrow \mathbb{R} & \text{strictly dominant linear functional} \end{cases}$

- **principal series** $\boxed{I_P(\delta,\nu)} = \mathrm{Ind}_{P=MAN}^{G}(\delta \otimes \nu \otimes triv)$

  G acts by left translation on:

  $\{F\colon G \to V^\delta \colon F|_K \in L^2,\ F(xman) = e^{-(\nu+\rho)log(a)}\delta(m)^{-1}F(x),\ \forall\, man \in P\}$

- **Langlands quotient** $\boxed{J(\delta,\nu)}$: 1! irred. quotient of $I_P(\delta,\nu)$

intertwining operator $A(\delta,\nu)\colon I_P(\delta,\nu) \longrightarrow I_{\bar{P}}(\delta,\nu),\ F \mapsto \int_{\bar{N}} F(x\bar{n})\, d\bar{n}$

$$J(\delta,\nu) \equiv \frac{I_P(\delta,\nu)}{\text{Ker } A(\delta,\nu)}$$

The Hermitian form on $J(\delta,\nu)$ is induced by the operator $A(\delta,\nu)$

A more informal definition (for real split groups)

Langlands Quotients $J(\delta,\nu)$ = **Candidates for Unitarity**

$\uparrow$

irreducible repr.s of G parameterized by:

- $\boxed{P} = MAN$: *a (fixed) minimal parabolic subgroup of G*
- $\boxed{\delta}$: *an irreducible representation of the finite group M*
- $\boxed{\nu}$: *an element of a cone, in a vector space of dim.=rank(G)*

If $J(\delta,\nu)$ is Hermitian, the form is induced by an operator $A(\delta,\nu)$.
Unitary dual Problem: finding all δ and ν s.t.

$$\boxed{J(\delta,\nu) \text{ is unitary}} \Leftrightarrow \boxed{A(\delta,\nu) \text{ is pos. semidefinite}}$$

Hard Problem: *the set of unitary parameters is very small.*

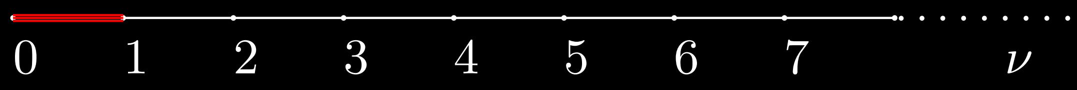
Spherical unitary duals of $SL(2,\mathbb{R})$, $SO(2,3)_0$ and $Sp(4)$

Fix $P = MAN$: minimal parabolic subgroup, δ: trivial repr. of M. The only parameter is ν; ν varies in a cone inside a vector space of dim.=rank(G). Consider the Langlands quotient $J(\nu) \equiv J(triv, \nu)$.

The values of ν s.t. $J(\nu)$ is unitary are painted in red:

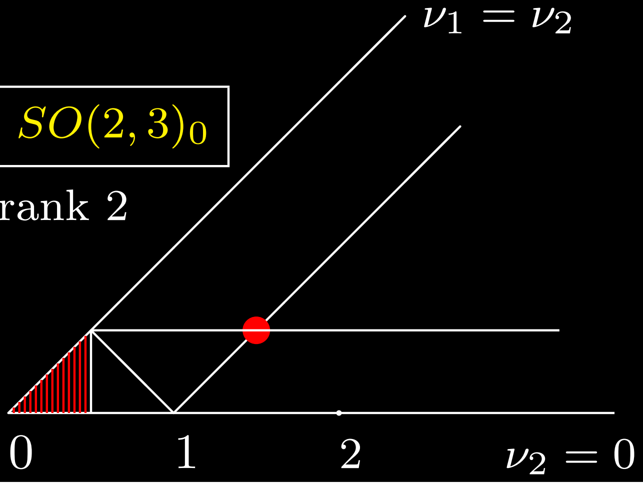
$SL(2,\mathbb{R})$

rank 1



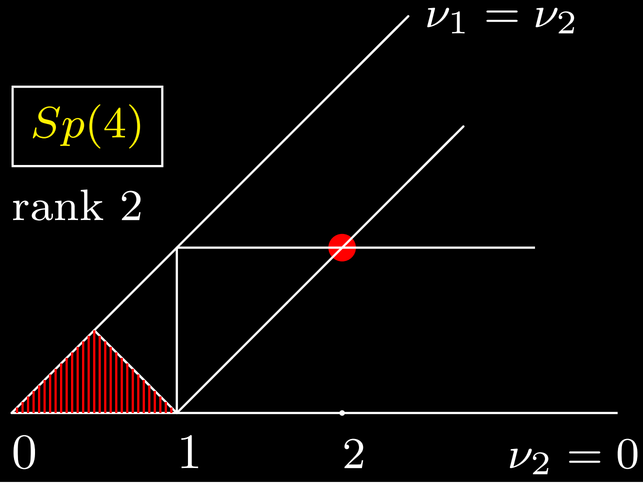
$SO(2,3)_0$

rank 2



$Sp(4)$

rank 2



PART 4

Find the Spherical Unitary Dual

i.e. discuss the unitarity of a spherical Langlands quotient $J(\nu)$

Spherical Unitary Dual of split groups

- **The setting:**

  G: a split connected real reductive group

  K: maximal compact subgroup $(G = SL(n,\mathbb{R}),\ K = SO(n))$

- **The problem:**

  Find the spherical unitary dual of G. *A representation of G is called* ***spherical*** *if it contains the trivial representation of K.*

- **Candidates:** spherical Langlands quotients $J(\nu) \equiv J(triv, \nu)$

  $\Rightarrow$ Equivalent problem: **Find all ν such that $J(\nu)$ is unitary**

- **Status quo:**

  By work of Knapp and Zuckermann, we know which $J(\nu)$'s are Hermitian, i.e have an invariant Hermitian form.

  If $J(\nu)$ is Hermitian, the form is induced by the operator $A(\nu)$.

  So $J(\nu)$ is unitary iff $A(\nu)$ is positive semidefinite.

Studying the signature of the operator $A(\nu)$

- The operator $A(\nu)$ acts on the spherical principal series, which is an infinite-dim.l vector space.
- $A(\nu)$ preserves the isotypic component of the various K-types $\mu \in \widehat{K}$ that appear in the spherical principal series ("spherical K-types").
- There are infinitely many spherical K-types, but each appears with finite multiplicity.
- Restricting the operator $A(\nu)$ to the isotypic component of μ, we get an operator $\boldsymbol{A_\mu(\nu)}$ for each spherical K-type $\boldsymbol{\mu}$.

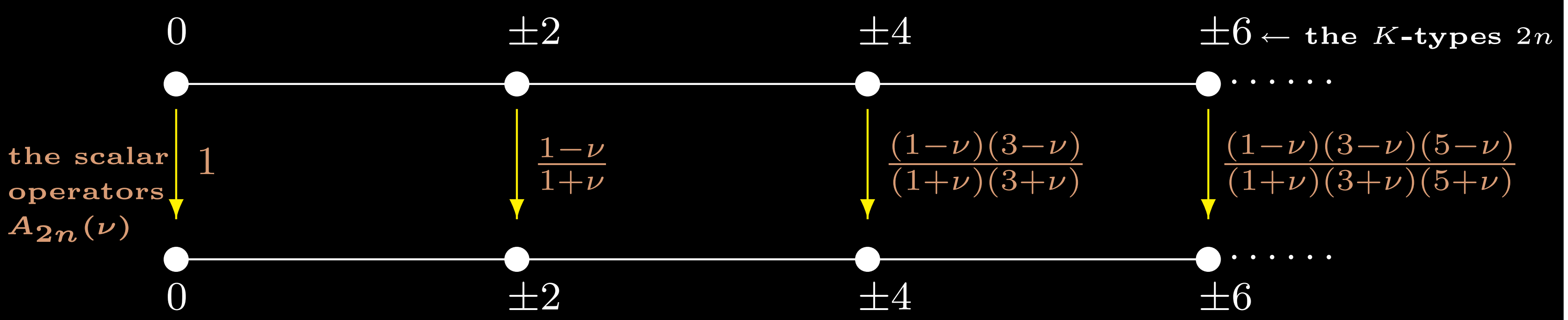
$J(\nu)$ is unitary iff $A_\mu(\nu)$ is positive semidefinite for all μ

The example of $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$, $K = SO(2,\mathbb{R})$, $\widehat{K} = \mathbb{Z}$, $\widehat{K}_{spherical} = 2\mathbb{Z}$

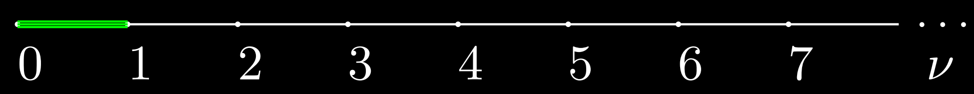
There is one operator $A_{2n}(\nu)$ for every even integer

Each operator $A_{2n}(\nu)$ acts by a scalar:



The representation $J(\nu)$ is unitary iff every $A_{2n}(\nu)$ is ≥ 0

$0 \leq \nu \leq 1$



Other real split reductive groups

- There are infintely-many spherical K-types μ
- For each μ, there is an operator $A_\mu(\nu)$
- The formula for $A_\mu(\nu)$ becomes very complicated if μ is "big"
- To obtain necessary and sufficient conditions for unitarity, one needs to study the signature of the operator $A_\mu(\nu)$ for all μ

Vogan, Barbasch: Only look for necessary conditions for unitarity

$\Downarrow$

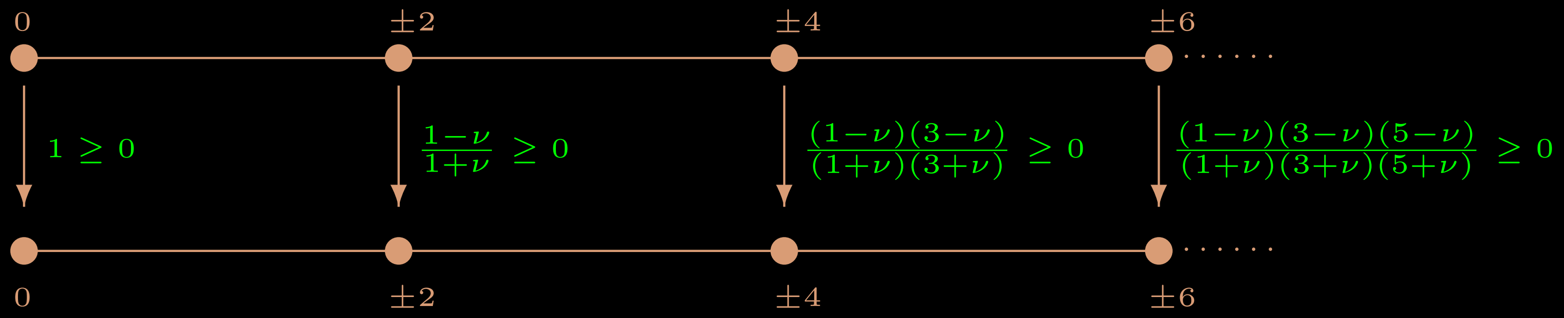
Isolate finitely many K-types μ (called "petite") s.t. the operator $A_\mu(\nu)$ is easy. Only compute the signature of $A_\mu(\nu)$ for μ petite.

Spherical Petite K-types for $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$, $\widehat{K}_{spherical} = 2\mathbb{Z}$. Spherical petite K-types: $n = 0, \pm 2$

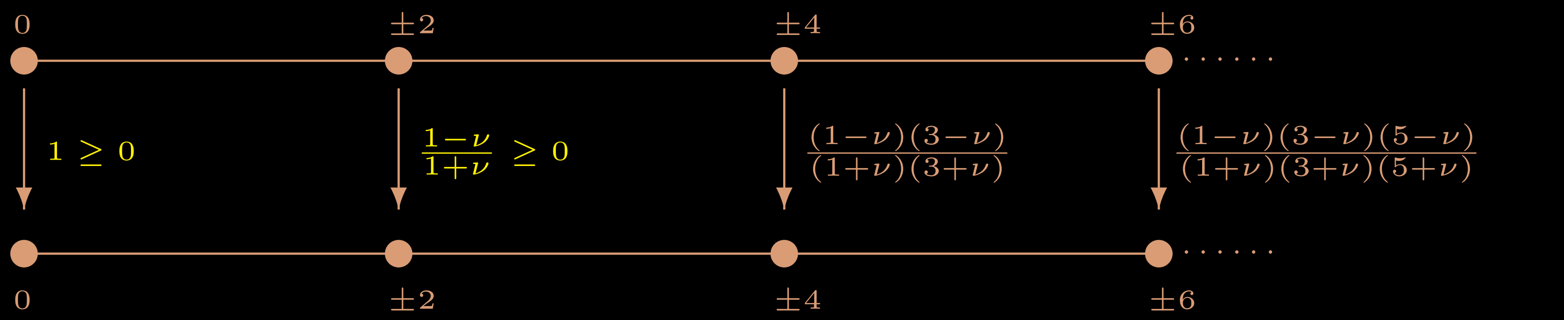
- **Necessary and Sufficient conditions for unitarity:**

$$A_{2n}(\nu) \textit{ is pos. semidefinite for every } K\textit{-type } 2n$$



- **Necessary conditions for unitarity:**

$$A_{2n}(\nu) \textit{ is pos. semidefinite for the petite } K\textit{-type } 2n = 0, \pm 2$$



Spherical Petite K-types for other split real groups

Definition [Barbasch, Vogan] For every root α, there is a subgroup $K_\alpha \simeq SO(2)$. A spherical K-type μ is called **petite** if the restriction of μ to K_α only contains the $SO(2)$-types 0 and ± 2.

If μ is petite, the intertwining operator $A_\mu(\nu)$ is "easy" to compute.

↑

?

↓

Easy : $A_\mu(\nu)$ behaves exactly like an operator for a p-adic group.

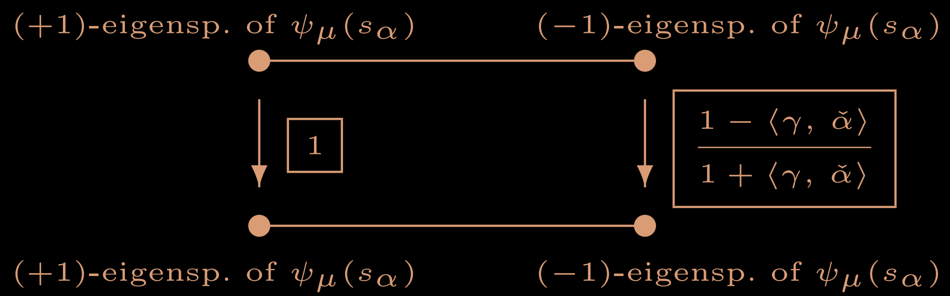
The operator $A_\mu(\nu)$ on a petite K-type μ

- $A_\mu(\nu)$ acts on the space $\mathrm{Hom}_M(\mu, \mathbb{C}) = (V_\mu^*)^M$.
- This space carries a representation $\boxed{\psi_\mu}$ of the Weyl group W.
- $A_\mu(\nu)$ only depends on the W-representation ψ_μ.

Indeed, we can compute $A_\mu(\nu)$ by means of Weyl group calculations:

$A_\mu(\nu)=\Pi_{\alpha \text{ simple}} A_\mu(s_\alpha, \gamma)$

$A_\mu(s_\alpha, \gamma)$ acts by $\rightsquigarrow$



For p-adic groups, there is an operator $A_\psi(\nu)$ for each W-type ψ.

μ petite $\Rightarrow$ $\boxed{\text{the real operator } A_\mu(\nu) = \text{the } p\text{-adic operator } A_{\psi_\mu}(\nu)}$

Comparing spherical unitary duals (real $\leftrightarrow$ p-adic)

- The unitarity of a Langlands quotient $J(\nu)$ for a split group G depends on the signature of some intertwining operators.
- For real groups there is an operator $A_\mu(\nu)$ for every irreducible representation μ of the maximal compact subgroup K.
- For p-adic groups there is an operator $A_\psi(\nu)$ for every irreducible representation ψ of the Weyl group W.

  It is enough to consider "relevant" W-types, because relevant W-types detect unitarity.
- *[Barbasch]* **Every relevant W-type is matched with a petite K-type s.t. the corresponding intertwining operators coincide.**
- This matching implies an inclusion of spherical unitary duals.

$J(\nu)_{\mathbb{R}}$ is unitary $=================>$ $J(\nu)_{\mathbb{Q}_p}$ is unitary

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$A_\mu(\nu) \geq 0$ $\forall \mu \in \widehat{K}_{sph.}$ $\Rightarrow$ $A_\mu(\nu) \geq 0$ $\forall \mu$ petite $\overset{(\star)}{\Rightarrow}$ $A_\psi(\nu) \geq 0$ $\forall \psi$ relevant $\overset{(\star)}{\Leftrightarrow}$ $A_\psi(\nu) \geq 0$ $\forall \psi \in \widehat{W}$

$(\star)$ For each relevant W-type ψ, there is a petite K-type μ s.t. the p-adic operator on ψ = the real operator on μ.

$(\star)$ Relevant W-types detect unitarity.

An embedding of spherical unitary duals for split groups

spherical unitary dual of $G(\mathbb{R})$	$\subseteq$ $\uparrow$	spherical unitary dual of $G(\mathbb{Q}_p)$

[Barbasch] : this inclusion is an equality for classical groups

The spherical unitary dual of a split p-adic group is known. Then

- **for classical real split groups, one obtains the full spherical unitary dual**
- **for non-classical real split groups, one obtains strong necessary conditions for the unitarity of a spherical Langlands quotient.**

PART 5

Find the Non-Spherical Unitary Dual

i.e. discuss the unitarity of a Langlands quotient $J(\delta, \nu)$, $\delta \neq triv$

Non-Spherical Unitary Dual

The non-spherical unitary dual of a real split group is mysterious

- Like in the spherical case, we need to understand which Hermitian Langlands quotients $J(\delta, \nu)$ are unitary

- To find necessary and sufficient conditions for unitarity, one needs to compute the signature of infinitely many operators.

  There is an operator $A_\mu(\delta, \nu)$ for every K-type μ containing δ.

  If μ is "big", computing $A_\mu(\delta, \nu)$ is extremely hard.

- Instead, we (only) look for necessary conditions...

Necessary conditions for unitarity

Spherical case - *Vogan, Barbasch:*

Define "spherical petite K-types", and use them to compare the spherical unitary dual of a real split group G with the spherical unitary dual of the corresponding p-adic group

Non-spherical case - *P., Barbasch:*

Define "non-spherical petite K-types", and use them to compare the non-spherical unitary dual of a real split group G with the spherical unitary dual of a (different) p-adic group

Non-spherical unitary dual

For each δ, we construct a p-adic group $G(\delta)_{\mathbb{Q}_p}$. Then we define non-spherical petite K-types (for δ) and we use them to compare

the non-spherical unitary dual of $G_{\mathbb{R}}$ induced by δ ?	with $<============>$	the spherical unitary dual of $G(\delta)_{\mathbb{Q}_p}$ ✓
$\Uparrow$		$\Uparrow$
candidates : $J(\delta,\nu)_{G_{\mathbb{R}}}$		*candidates* : $J(\nu_0)_{G(\delta)_{\mathbb{Q}_p}}$
$J(\delta,\nu)_{G_{\mathbb{R}}}$ unitary $\Leftrightarrow$ $\boxed{A_\mu(\delta,\nu)} \geq 0,\ \forall \mu \in \widehat{K}$		$J(\nu_0)_{G(\delta)_{\mathbb{Q}_p}}$ unitary $\Leftrightarrow$ $\boxed{A_\psi(\nu)} \geq 0,\ \forall \psi \in \widehat{(W_0)}_{rel}$

One doesn't always get an embedding of unitary duals

$J(\delta,\nu)$ is unit. for $G_{\mathbb{R}}$ $\overset{???}{======>}$ $J(triv,\nu_0)$ is unit. for $G(\delta)_{\mathbb{Q}_p}$

$\Updownarrow$ $\Updownarrow$

$A_\mu(\delta,\nu) \geq 0$ $\forall \mu \in \widehat{K}$ | $A_\psi(\nu_0) \geq 0$ $\forall \psi \in \widehat{W_0}$

$\Updownarrow$ $\Updownarrow$

$A_\mu(\delta,\nu) \geq 0$ $\forall \mu$ petite $\overset{???}{==================>}$ $A_\psi(\nu_0) \geq 0$ $\forall \psi$ relev

if you have enough petite K-types

to match all the relevant W_0-types

The linear split group F_4

- $G = F_4$
- $K = [Sp(1) \times Sp(3)]/\{\pm I\}$

  K-types= irreducible repr.s of K, classified by highest weight:
  $\mu = (a_1|a_2,\, a_3,\, a_4)$, with $a_1 \geq 0$, $a_2 \geq a_3 \geq a_4 \geq 0$, $\sum a_i \equiv 0\,(2)$

Minimal Principal Series : $I(\delta, \nu)$

- $P = MAN$= a minimal parabolic subgroup
- M: a finite abelian group of order 16
- A: vector group ($\dim Lie(A) = 4$)
- δ : irreducible representation of M
- ν : dominant linear functional on $Lie(A)$

Problem: discuss the unitarity of the Langlands quotients $J(\delta, \nu)$

The Weyl group W acts on $\widehat{M}$. Let $W(\delta)$ be the stabilizer of δ.

- $W(\delta)$ only depends on the W-orbit of δ
- $W(\delta)$ is the Weyl group of a root system $\Delta_0(\delta)$.
  Let $G(\delta)$ be the corresponding split group.

representative for the W-orbit of δ	**root system $\Delta_0(\delta)$**	**corresponding split group $G(\delta)$**
δ_1	F_4	F_4
δ_3	C_4	$Sp(4)$
δ_{12}	B_3A_1	$SO(4,3)_o \times SL(2)$

Using petite K-types, we relate the unitarity of a (possibly non-spherical) Langlands quotient of G induced from δ to the unitarity of a spherical Langlands quotient of $G(\delta)$

Examples for the linear split group F_4

- $\delta = \delta_{12}$; $G(\delta) = SO(4,3)_0 \times SL(2)$
- Every relevant W-type for $G(\delta)$ can be matched with a petite K-type for F_4. Hence there is an inclusion of unitary duals:

$$\boxed{\begin{array}{c}\text{unitary parameters}\\ \text{for } (\delta_{12}, F_4)\end{array}} \subset \boxed{\begin{array}{c}\text{unitary parameters}\\ \text{for } (triv, SO(4,3)_0 \times SL(2))\end{array}}$$

- $\delta = \delta_3$; $G(\delta) = Sp(4)$
- We can match every relevant W-type for $G(\delta)$ *except* $\mathbf{1} \times \mathbf{3}$. Hence we obtain a weaker inclusion:

$$\boxed{\begin{array}{c}\text{unitary}\\ \text{parameters}\\ \text{for } (\delta_3, F_4)\end{array}} \subseteq \boxed{\boxed{\begin{array}{c}\text{unitary}\\ \text{parameters}\\ \text{for } (triv, Sp(4))\end{array}} \cup \boxed{\begin{array}{c}\text{non-unitarity region}\\ \text{for (triv, } Sp(4))\\ \text{ruled out by } 1 \times 3\end{array}}}$$

Conclusions

A Hermitian representation is unitary if and only if the invariant Hermitian form is positive definite.

Using petite K-types, we compare invariant forms on Hermitian representations for real and p-adic groups.

This comparison leads to interesting relations between the unitary duals of the two groups.

For example, it implies that the spherical unitary dual of a real split group is always contained in the spherical unitary dual of the corresponding p-adic group.

In the non-spherical case, you still get very interesting inclusions.