Dynamical Systems Math 117

Final Exam

This is a take home exam. Problems marked by a star can be solved for extra credit. You can (and should) use the textbook, lecture notes, online sources, but not a direct help of other people.

Problem 1. *(Homeomorphisms of the circle)*

a) Let f, g be homeomorphisms of the circle. Assume that each of them has exactly one fixed point. Prove that f and g are topologically conjugate.

b) How many essentially different (i.e. not topologically conjugate) orientation-preserving homeomorphisms of the circle are there among those that have exactly two fixed points? Exactly three fixed points? Exactly four fixed points?

*c)** Fix $\alpha \notin \mathbb{Q}$, and consider the set of all homeomorphisms of the circle with rotation number α. How many equivalence classes (with respect to the topological conjugacy) are there? Finitely many? Countably many? Uncountably many?

Problem 2. *(Piecewise linear maps of the interval)*

Periodic orbits of what (smallest) period does the map $f : [0,1] \to [0,1]$ have?

a) $f(x) = \begin{cases} 2x, & \text{if } x \in [0, 0.5]; \\ 1.5 - x, & \text{if } x \in [0.5, 1]. \end{cases}$

b) $f(x) = \begin{cases} 0.5 + x, & \text{if } x \in [0, 0.5]; \\ 2 - 2x, & \text{if } x \in [0.5, 1]. \end{cases}$

c) $f(x) = \begin{cases} 1 - 2x, & \text{if } x \in [0, 0.5]; \\ x - 0.5, & \text{if } x \in [0.5, 1]. \end{cases}$

$d)^*$ $f(x) = \begin{cases} x, & \text{if } x \in [0, 0.5]; \\ 2x - 0.5, & \text{if } x \in [0.5, 0.75]; \\ 2.5 - 2x, & \text{if } x \in [0.75, 1]. \end{cases}$

Problem 3. *(Symbolic dynamical systems)*

Let Σ_3 be the space of all bi-infinite sequences of $0, 1, 2$, and $\sigma : \Sigma_3 \to \Sigma_3$ be the topological Bernoulli shift. Let $\Omega \subseteq \Sigma_3$ be the subset of sequences such that the sum of any three subsequent elements is even.

$a)$ Show that Ω is an invariant set of the topological Bernoulli shift.

$b)$ Is the map $\sigma : \Omega \to \Omega$ transitive?

$c)$ Find the number of periodic points in Ω of period 5.

$d)^*$ Find the topological entropy of $\sigma : \Omega \to \Omega$.