Dynamical Systems Math 117

Midterm Exam

This is a take home exam. Problems marked by a star can be solved for extra credit. You can (and should) use the textbook, lecture notes, online sources etc., but not a direct help of other people.

Problem 1. *(3D Lissajous curves)*

Consider a curve $\gamma : \mathbb{R}^1 \to \mathbb{R}^3$,

$$\gamma(t) = (A\cos(\omega_1 t + a), B\cos(\omega_2 t + b), C\cos(\omega_3 t + c)),$$

where $A, B, C > 0$, $\omega_1, \omega_2, \omega_3 \neq 0$.

a) Prove that the curve γ is closed if and only if there are rational numbers p_1, p_2, p_3 such that $\omega_1 : p_1 = \omega_2 : p_2 = \omega_3 : p_3$.

b)* Under what condition on $\omega_1, \omega_2, \omega_3$ the curve γ is dense in $[-A, A] \times [-B, B] \times [-C, C]$?

c)* Can you describe the behavior of the n-dimensional analog of Lissajous curves in $\mathbb{R}^n$ depending on the frequencies $\omega_1, \ldots, \omega_n$?

Problem 2. *(Products of dynamical systems)*

Suppose X, Y are metric spaces, and $f : X \to X$, $g : Y \to Y$ are continuous maps. The product of f and g is the map $f \times g : X \times Y \to X \times Y$ defines by

$$f \times g(x, y) = (f(x), g(y)).$$

a) TRUE or FALSE: If both f, g are minimal, then $f \times g$ is also minimal.

b) TRUE or FALSE: If both f, g are transitive, then $f \times g$ is also transitive.

c) TRUE or FALSE: If both f, g have dense sets of periodic points, then $f \times g$ has the same property.

$d)$ TRUE or FALSE: If f has infinitely many periodic points, then $f \times g$ must have infinitely many periodic points.

$e)^*$ TRUE or FALSE: For any $x \in X$, $y \in Y$, we have $\omega_{f\times g}((x,y)) = \omega_f(x) \times \omega_g(y)$.

$f)^*$ TRUE or FALSE: If $p \in \mathbb{N}$, and neither f nor g have periodic orbits of (smallest) period p, then $f \times g$ does not have periodic orbits of period p either. *(Hint: Does the answer depend on p?)*

<u>Problem 3.</u> *(Dynamical systems with infinite entropy and the Gauss map)*

Let $G : [0,1] \to [0,1]$ be defined as

$$G(x) = \begin{cases} \frac{1}{x} \pmod 1, & \text{if } x \in (0,1]; \\ 0, & \text{if } x = 0. \end{cases}$$

This map is usually called the Gauss map.

$a)$ Sketch the graph of the Gauss map.

$b)$ Describe as many properties of the Gauss map as you can (study the fixed and periodic points, transitivity, etc.).

$c)$ Show that if $x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$ (this expression is called *continued fraction expansion*), where $a_i \in \mathbb{N}$, then $G(x) = \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \dots}}}}$.

$d)$ What is the topological entropy of the Gauss map? Explain your answer.

$e)^*$ Prove that for any $\rho \in [0, +\infty]$ there exists an invariant subset $S_\rho \subset [0,1]$ such that the restriction of the Gauss map to S_ρ has topological entropy ρ.