LINEAR ALGEBRA MATH 121A

Final Exam (answers)

Problem 1.

Which of the following subsets of $P_3(\mathbb{R})$ are subspaces? Explain your answers.

a) $U_1 = \{p \in P_3(\mathbb{R}) \mid p(-1) = p(1)\}$

b) $U_2 = \{p \in P_3(\mathbb{R}) \mid p(0) = 0\}$

c) $U_3 = \{p \in P_3(\mathbb{R}) \mid p(4) = 3p(-2) - 3\}$

d) $U_4 = \{p \in P_3(\mathbb{R}) \mid p(1) + p'(2) = 0\}$

e) $U_5 = \{p \in P_3(\mathbb{R}) \mid |p(0)| = |p'(0)|\}$

Answer: a), b), d) - subspaces; c) e) - not subspaces.

Problem 2.

Let V be a finite dimensional vector space, and $U, W \subseteq V$ be subspaces. Prove that

$$\dim(U \cap W) = \dim U$$

if and only if $U \subseteq W$.

Problem 3.

Let $U \subset P(\mathbb{R})$ be defined as

$$U = \text{span}(1 + x + x^2 + x^3, x^3 - x^2 + x - 1, 2x^3 + x^2 + 2x + 1, x^3 - 3x^2 + x - 3).$$

Find $\dim U$.

Answer: $\dim U = 2$. Notice that $\{x^3 + x, x^2 + 1\}$ is a basis in U.

Problem 4.

Let us denote by $M_{n\times n}(\mathbb{R})$ the set of $n \times n$ matrices with real entries, and by $\mathcal{L}(\mathbb{R}^n)$ the space of linear operators on $\mathbb{R}^n$. For each of the following statements answer whether it is true or false (explain your answer):

a) If rank of $A \in M_{n\times n}(\mathbb{R})$ is equal to n, then rank of A^2 is also equal to n.

b) If $A \in M_{n\times n}(\mathbb{R})$ is an upper-triangular matrix, then its rank is equal to the number of non-zero elements on the diagonal.

c) If $A \in M_{n\times n}(\mathbb{R})$ is a diagonal matrix, then its rank is equal to the number of non-zero elements on the diagonal.

d) If $T \in \mathcal{L}(\mathbb{R}^n)$, and $\dim range(T) < n$, then for sufficiently large $m \in \mathbb{N}$ we must have $T^m = 0$.

e) For any operators $T, S \in \mathcal{L}(\mathbb{R}^n)$, dimension of $range(ST)$ cannot be larger than dimension of $range(T)$.

Answer: a) - true, b) - false, c) - true, d) - false, e) - true.

Problem 5.

Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + be^x$, $a, b \in \mathbb{R}$. Consider the transformation

$$T : f \mapsto f + 2e^x f' + \left(4xe^{-x} - 2e^x + 2\right) f''$$

a) Prove that T is a linear operator on V.

b) Find eigenvalues of $T : V \to V$.

c) Find eigenvectors of $T : V \to V$.

Answer: Eigenvalues $\{5, -1\}$. Eigenvectors $f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$.