INTRODUCTION TO TOPOLOGY, MATH 141, HW#1

Problem 1.

Consider the space X of all sequences of 0's and 1's, i.e.

$$X = \{\omega_1\omega_2\dots \mid \omega_i \in \{0,1\}\ \}.$$

If $x = x_1x_2x_3\dots$ and $y = y_1y_2y_3\dots$, set

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2^{-n}, & \text{if } x_n \neq y_n \text{ and } x_i = y_i \text{ for } i < n. \end{cases}$$

Prove that (X, d) is a metric space.

Problem 2.

Suppose that a metric space (X, d) contains exactly three points, $X = \{x_1, x_2, x_3\}$. Is it true that one can find three points $p_1, p_2, p_3 \in \mathbb{R}^2$ such that for any $i, j \in \{1, 2, 3\}$

$$dist(p_i, p_j) = d(x_i, x_j)?$$

In other words, is it possible to find a "copy" of the metric space (X, d) as a subset of the plane?

Problem 3.

Suppose that a metric space (X, d) contains exactly four points, $X = \{x_1, x_2, x_3, x_4\}$. Is it true that one can find four points p_1, p_2, p_3, p_4 in some Euclidian space $\mathbb{R}^n$ such that for any $i, j \in \{1, 2, 3, 4\}$

$$dist(p_i, p_j) = d(x_i, x_j)?$$

Problem 4.

Let (X, d) be a metric space, and $Y \subseteq X$ be a subset. A point $x \in X$ is called a boundary point of Y is for any open ball $B(x, r)$ we have

$$Y \cap B(x,r) \neq \emptyset, \text{ and } (X \backslash Y) \cap B(x,r) \neq \emptyset.$$

We will call the set of all boundary points of Y *the boundary of* Y (denoted ∂Y). Prove that for any set Y the boundary ∂Y is a closed set.

Problem 5.

Let (X,d) be a metric space. Let us denote by $S(x,r)$ the sphere or radius $r>0$ around the point $x \in X$, $S(x,r) = \{y \in X \mid d(x,y) = r\}$. Is it true that for any open ball $B(x,r)$ its boundary must be $S(x,r)$?

Problem 6.

Suppose that $d : X \times X \to \mathbb{R}$ is a distance function.

a) Is it true that $\rho : X \times X \to \mathbb{R}$, $\rho(x,y) = (d(x,y))^2$ is also a distance function?

b) Is it true that $\rho : X \times X \to \mathbb{R}$, $\rho(x,y) = \sqrt{d(x,y)}$ is also a distance function?

c) Is it true that $\rho : X \times X \to \mathbb{R}$, $\rho(x,y) = 3d(x,y)$ is also a distance function?

d) Is it true that $\rho : X \times X \to \mathbb{R}$, $\rho(x,y) = \sqrt{d(x,y)} + 2d(x,y)$ is also a distance function?

Explain your answers (i.e. prove the statement or give a counterexample).

Problem 7.

Let X be the set of all closed bounded subsets on a plane $\mathbb{R}^2$. For any $A, B \in X$ define

$$d(A,B) = \max\left\{\sup_{a\in A} \text{dist}(a,B),\ \sup_{b\in B} \text{dist}(b,A)\right\}.$$

Prove that (X,d) is a metric space.
Remark: The metric d defined in this way is called Hausdorff metric.