Introduction to Topology, Math 141, HW#3

Problem 1.

Determine the interior, the closure, the boundary, and the set of isolated points of each of the following subsets of $\mathbb{R}^2$ (explain your answer; you can supplement the answer by the picture):

a) $\{(x,y) \mid x^2+y^2 \in (1-\frac{1}{2n}, 1-\frac{1}{2n+1})$ for some $n \in \mathbb{N}\}$

b) $\{(x,y) \mid x \in \mathbb{Q},\ y \notin \mathbb{Q}\}$

c) $\{(x,y) \mid x = \frac{1}{n}$ for some $n \in \mathbb{N}\} \bigcap \{(x,y) \mid x+y = \frac{1}{n}$ for some $n \in \mathbb{N}\}$

Problem 2.

For each of the metrics $d_{\sqrt{}}$, d_{max}, and d_+ in $\mathbb{R}^2$ draw the set

$$\{x \in \mathbb{R}^2 \mid d(x,x_0) + d(x,x_1) = d(x_0,x_1)\},$$

where $x_0 = (0,0)$, $x_1 = (1,1)$.

Problem 3.

Show that

$$d((x_1,x_2,x_3),(y_1,y_2,y_3)) = |x_1-y_1| + \max(|x_2-y_2|, |x_3-y_3|)$$

is a metric in $\mathbb{R}^3$. Draw the unit ball with respect to this metric.

Problem 4.

A collection of subsets of a set is said to have the *finite intersection property* if each finite subcollection has a non-empty intersection. Prove that a metric space (X,d) is compact (i.e. every open cover has a finite subcover) if and only if every collection of closed sets in X with the finite intersection property has a nonempty intersection.

Problem 5.

Which of the following subsets in $\mathbb{R}^1$ are compact?

a) $\{0\} \cup \{1/n\}_{n=1}^{\infty}$

b) $\cup_{n \in \mathbb{N}} [2n, 2n+1]$

c) $\cup_{n \in \mathbb{N}} [\frac{1}{2n+1}, \frac{1}{2n}]$

Problem 6.

Suppose (X, d) is a compact metric space, and $f : X \to \mathbb{R}$ is a function such that for every point $x \in X$ there is $r(x) > 0$ and $\delta(x) > 0$ such that for every $y \in B(x, r(x))$ we have $f(y) > \delta(x)$. Prove that there exists $\varepsilon > 0$ such that for every $y \in X$ one has $f(y) > \varepsilon$.

Problem 7.

Prove that a discrete metric space is compact if and only if it is finite.