INTRODUCTION TO TOPOLOGY, MATH 141, HW#4

Problem 1.

Let (X,d) be a metric space, and $z \in X$ be a point in X. Prove that the function

$$f : X \to \mathbb{R}, \quad f(x) = d(x,z),$$

is uniformly continuous.

Problem 2.

The *Hilbert cube* H^∞ is a collection of all real sequences $x = \{x_n\}_{n\in\mathbb{N}}$ with $|x_n| \le 1$ for $n = 1, 2, \ldots,$.

a) Show that $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞;

b) Is (H^∞, d) compact?

Problem 3.

Suppose (X,d) is a metric space and $f : X \to \mathbb{R}$ is continuous. Prove that $\{x \in X \mid f(x) = 0\}$ is a closed set.

Problem 4.

For each of the following functions determine whether it is continuous. If yes, is it uniformly continuous? Explain your answers.

a) $f : \mathbb{Q} \to \mathbb{Q}$,

$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ \frac{1}{m}, & \text{if } x = \frac{n}{m} \neq 0, n \in \mathbb{Z}, m \in \mathbb{N}, \text{ and } m \text{ and } n \text{ are relatively prime.} \end{cases}$

b) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$.

c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_1, x_2)$.

Problem 5.

Let X be the space of real valued continuous functions on $[0,1]$. Prove that

$$\|f\| = \left(\int_0^1 f^2(x)dx \right)^{1/2}$$

defines a norm in X.

Problem 6.

Let X be the space of real valued continuous functions on $[0, 1]$. Which of the following formulas define a norm on X?

a) $\|f\| = \int_0^1 x|f(x)|dx$

b) $\max_{x\in[0,1]} f^2(x)$

c) $|f(0)| + |f(1)|$

d) $|f(0)| + \int_0^1 |f(x)|dx$

Problem 7.

Let X be the space of all real polynomials. Which of the following formulas define a norm on X?

a) $\|p\| = \int_0^1 |p(x)|dx$

b) $\|p\| = \int_2^3 |p(x)|dx$

c) $\|p\| = \sum_{i=1}^{\infty} 2^{-i}|p(i)|$

d) $\|p\| = \sum_{n=0}^{\infty} |p^{(n)}(0)|$, where $p^{(n)}$ is n-th derivative of p; in particular, $p^{(0)} = p$.