Introduction to Topology, Math 141, HW#5

Problem 1.

Let us recall that $l^\infty = \{\bar{x} = (x_1, x_2, \ldots, x_n, \ldots) \mid \{x_n\}_{n\in\mathbb{N}} \text{ is bounded } \}$. Fix $k \geq 1$ and define $T : l^\infty \to \mathbb{R}$ by $T(\bar{x}) = x_k$. Show that T is linear, and has $\|T\| = 1$.

Problem 2.

Let us recall that

$$l^1 = \{\bar{x} = (x_1, x_2, \ldots, x_n, \ldots) \mid \sum_{n=1}^{\infty} |x_n| < \infty\},$$

$$l^2 = \{\bar{x} = (x_1, x_2, \ldots, x_n, \ldots) \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Define a linear map $T : l^2 \to l^1$ by $T(\bar{x}) = \bar{y}$, where $y_n = \frac{x_n}{n}$. Is T bounded? If so, what is $\|T\|$?

Problem 3.

Let $C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$, with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Consider the linear map $T : C[0,1] \to \mathbb{R}$ given by $T(f) = \int_0^1 f(x)dx$. What is $\|T\|$?

Problem 4.

Let X be the ray $[0, +\infty)$, and let $\mathcal{T}$ consists of $\emptyset$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that $\mathcal{T}$ is a topology.

Problem 5.

List all topological structures in a two-point set.

Problem 6.

Let X be $\mathbb{R}$, and let $\mathcal{T}$ consists of the empty set and all infinite subsets of $\mathbb{R}$. Is $\mathcal{T}$ a topology?

Problem 7.

Let X consists of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topologies:

1) $\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}$;

2) $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}$;

3) $\emptyset, X, \{a, c, d\}, \{b, c, d\}$?