Introduction to Topology, Math 141, Practice Problems

Problem 1.

Give an example of a non-metrizable topological space. Explain.

Problem 2.

Introduce a topology on $\mathbb{N}$ by declaring that open sets are $\emptyset, \mathbb{N}$, and all sets that can be represented as unions of (infinite) arithmetic progressions. Check that this is indeed a topological space, and prove that any finite set is closed. Is it true that any closed set is finite?

Problem 3.

Let (X, d) be a metric space. Find out (i.e. prove or give a counterexample) whether it is **TRUE or FALSE** that for any subsets $A, B \subset X$ one has

a) $\text{int}\,(A \cup B) = \text{int}\,A \cup \text{int}\,B$

b) $\text{int}\,(A \cap B) = \text{int}\,A \cap \text{int}\,B$

c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

e) $\overline{A \backslash B} = \overline{A} \backslash \overline{B}$

f) $\overline{A \backslash B} = \overline{A} \backslash \text{int}\,(B)$

g) $\overline{\text{int}\,A} = \text{int}\,(\overline{A})$

h) $\text{int}\,(\overline{\text{int}\,A}) = \text{int}\,A$

i) $\text{int}\,(\text{int}\,A) = \text{int}\,A$

j) $\overline{\text{int}\,(\overline{A})} = \overline{A}$

k) $\overline{A}\backslash\text{int}\,(A) = \overline{(X\backslash A)}\backslash(\text{int}\,(X\backslash A))$

Problem 4.

Let $X = \{a, b\}$ be a two-point set. Check that $\mathcal{T} = \{\emptyset, X, \{a\}\}$ is a topology on X. Is $(X, \mathcal{T})$ a Hausdorff topological space?

Problem 5.

Which of the following metric spaces are complete? Bounded? Totally bounded?

a) $X = \mathbb{R},\ \ d(x, y) = |x - y|$

b) $X = \mathbb{Q},\ \ d(x, y) = |x - y|$

c) $X = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\},\ \ d(x, y) = |\log x - \log y|$

d) $X = (-1, 1),\ \ d(x, y) = |\tan(\frac{\pi x}{2}) - \tan(\frac{\pi y}{2})|$

e) $X = \mathbb{Z},\ \ d(m, n) = |n - m|$

f) $X = \mathbb{N},\ \ d(m, n) = |\frac{1}{m} - \frac{1}{n}|$

Problem 6.

TRUE or FALSE:

a) closed subset of a compact metric space is compact;

b) compact subset of a metric space is closed;

c) one-point subset of any metric space is compact;

d) one-point subset of any topological space is compact;

e) any finite subset of a topological space is closed.

Problem 7.

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ be a uniformly continuous function. Which of the following statements is correct? Explain.

a) If (X, d) is bounded, then $f(X)$ is bounded.

b) If (X, d) is totally bounded, then $f(X)$ is totally bounded.

Problem 8.

Let X be the normed linear space of continuous real valued functions on $[0, 1]$, with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0, & \text{if } x \in [1, 1/2); \\ 1, & \text{if } x \in [1/2, 1]. \end{cases}$$

Show that the linear map $T : X \to X, f \mapsto gf$, is continuous, and find its norm.

Problem 9.

Prove that a linear operator from a normed linear space X into a normed linear space Y is bounded if and only if it maps bounded sets onto bounded sets.

Problem 10.

Suppose that a linear operator $T : X \to Y$ from a normed linear space X into a normed linear space Y is bounded and invertible. Is it true that the inverse $T^{-1} : Y \to X$ also must be bounded? Prove or give a counterexample.

Problem 11.

Suppose that X is an infinite set equipped with the cofinite topology.

(1) Let $A \subset X$ be a finite set. Compute $\bar{A}$.

(2) Let $A \subset X$ be an infinite set. Compute $\bar{A}$.

Problem 12.

(1) Let X be a non-empty set equipped with the discrete topology. Show

that X is compact if and only if X is finite.

(2) Let X be a topological space, and let Y and Z be compact subsets of X. Show that $Y \cup Z$ is compact.

Problem 13.

Suppose that X is a topological space and that A, B are connected subsets of X such that $A \cap \bar{B} \neq \emptyset$. Show that $A \cup B$ is connected.

Problem 14.

Is it true that every bounded metric space must be separable? Prove or give a counterexample.

Problem 15.

Consider the map $f : [1, +\infty) \to [1, +\infty)$, $f(x) = x + \frac{1}{x}$. Show that

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y$. Does f have any fixed points? Why does not it contradict to the Contraction Mapping Principle?

Problem 16.

Suppose X is a metric space, $A, B \subset X$, and functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are continuous and agree on $A \cap B$ (i.e. if $x \in A \cap B$, then $f(x) = g(x)$). Prove that the function $h : A \cup B \to \mathbb{R}$,

$$h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B, \end{cases}$$

is continuous on $A \cup B$.

Problem 17.

Let X, Y be metric spaces. Which of the following statements is correct? Explain.

1) A continuous function $f : X \to Y$ maps Cauchy sequences into Cauchy sequences.

2) A uniformly continuous function $f : X \to Y$ maps Cauchy sequences into Cauchy sequences.

Problem 18.

Suppose that $\{U_n\}_{n\in\mathbb{N}}$, $U_n \subset \mathbb{R}$, is a countable collection of pairwise disjoint open intervals.

1) Is it true that $\partial(\overline{\cup_{n=1}^{\infty} U_n}) = \cup_{n=1}^{\infty}(\partial U_n)$?

2) Is it true that $\partial(\overline{\cup_{n=1}^{\infty} U_n})$ is countable?

Problem 19.

Which of the following collections of subsets of $\mathbb{R}^2$ form a base of a topology (not necessarily of standard topology) on $\mathbb{R}^2$?

1) Collection of all open discs.

2) Collection of all open squares.

3) Collection of all open triangles.

4) Collection of all open half-planes.

5) Collection of all open sectors between some rays coming from the origin.

6) Collection of all open discs of radius one.

7) Collection of all one-point sets.

8) Collection of all sets whose complement is a finite union of lines.

Problem 20.

Let (X, d) and (Y, ρ) be metric spaces. Which of the following define a metric in $X \times Y$?

a) $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$

b) $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2)^2 + \rho(y_1, y_2)^2$

c) $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)} + \sqrt{\rho(y_1, y_2)}$

d) $d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)+\rho(y_1,y_2)}$