REAL ANALYSIS
MATH 205A & MATH H140A, FALL 2015

Final Sample

Problem 1.

Prove that the function $f(x) = \sin\sqrt{x}$ is uniformly continuous on $[0, +\infty)$.

Problem 2.

Let (M, d) be a metric space, and $E \subset M$ be a compact set. Prove that E is a closed subset of M.

Problem 3.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Set $f_n(x) = f\left(x + \frac{1}{n}\right)$.

a) Is it true that the sequence $\{f_n\}$ converges to f pointwise?

b) Is it true that the sequence $\{f_n\}$ converges to f uniformly?

Problem 4.

Suppose that M is a compact metric space without isolated points, and $f : M \to M$ is a homeomorphism. Suppose that there exists a point $x_0 \in M$ such that the orbit of x_0 under the action of f (i.e. the set $\{f^n(x_0)\}_{n\in\mathbb{Z}}$, where $f^0(x_0) = x_0$, $f^n(x_0) = f(f^{n-1}(x_0))$ for $n > 0$, and $f^n(x_0) = f^{-1}(f^{n+1}(x_0))$ for $n < 0$) is dense in M. Prove that M contains uncountably many points with dense orbits.

Problem 5.

Suppose (M, d) is a compact metric space, and $f : M \to M$ is a continuous mao such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in M$. Prove that f has a unique fixed point in M.