Midterm Sample

Problem 1.

Prove that the following improper integral converges: \( \int_0^\infty \frac{\cos(2x)}{x^{1/3}} \, dx \)

Problem 2.

Suppose \( U \) is an open subset of \( \mathbb{R} \), containing a point \( x_0 \), \( f \) and \( g \) are real-valued functions, defined on \( U \), such that \( g \) is continuous, \( f \) is differentiable, and \( f(x_0) = 0 \). Prove that the product \( fg \) is differentiable at \( x_0 \).

Problem 3.

Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is such that \( f, f', \) and \( f'' \) are all bounded and continuous real-valued functions. Assume also that \( f(0) = f'(0) = 0 \). Analyze the convergence of the series \( \sum_{n=1}^{\infty} f(\frac{x}{n}) \), i.e. determine for which values of \( x \) the series is convergent, and for which values of \( x \) the series converges absolutely. Is the convergence uniform? Justify all your claims.

Problem 4.

Determine whether the family of functions \( f_\alpha : [0, 1] \to \mathbb{R} \), given by the formula
\[
f_\alpha(x) = \frac{1}{1 + e^{\alpha x}} \quad \text{for} \quad x \in [0, 1], \alpha \in [1, \infty),
\]
is equicontinuous on \([0, 1]\). Justify your answer.

Problem 5.

Let \( f_n, n = 1, 2, \ldots \) and \( f \) be Riemann integrable real-valued functions defined on \([0, 1]\). For each of the following statements, determine whether the statement is true or not:

(a) If \( \lim \limits_{n \to \infty} \int_0^1 |f_n(x) - f(x)| \, dx = 0 \), then \( \lim \limits_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^2 \, dx = 0 \).

(b) If \( \lim \limits_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^2 \, dx = 0 \), then \( \lim \limits_{n \to \infty} \int_0^1 |f_n(x) - f(x)| \, dx = 0 \).