### Practice Exam

Friday, May 25, 2012 — 1:00pm-3:30pm

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Student’s name:
Problem 1.

TRUE or FALSE? For any bounded collection of real numbers \( \{a_{mn}\}_{m,n \in \mathbb{N}} \) we have

\[
\limsup_{n \to \infty} \left( \limsup_{m \to \infty} a_{mn} \right) = \limsup_{m \to \infty} \left( \limsup_{n \to \infty} a_{mn} \right)
\]
Problem 2.

Which number is larger, $\pi^3$ or $3\pi$?
Problem 3.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous for $x \in \mathbb{R}$, differentiable for $x \in (-\infty, 0) \cup (0, \infty)$, and $\lim_{x \to 0} f'(x) = 2$. Show that $f$ is differentiable at $x = 0$ and $f'(0) = 2$. 
Problem 4.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables and assume that the restriction of $f$ to any line in $\mathbb{R}^2$ is differentiable (i.e. for any $a, b, c, d \in \mathbb{R}$ the function of one variable $g(t) = f(at + b, ct + d)$ is differentiable). Is $f$ continuous on $\mathbb{R}^2$? Prove or give a counterexample.
Problem 5.

Suppose \((x_0, y_0, u_0, v_0)\) is a solution to the system of equations
\[
\begin{align*}
    e^x \cos y + 2u - v &= 0 \\
    e^x \sin y - u + 2v &= 0
\end{align*}
\]

(a) Show that in a neighborhood of \((x_0, y_0, u_0, v_0)\) the system can be solved for \((x, y)\) as a function of \((u, v)\);

(b) Show that in a neighborhood of \((x_0, y_0, u_0, v_0)\) the system can be solved for \((u, v)\) as a function of \((x, y)\).
Problem 6.

Suppose \( \{x_n\} \) is a sequence in a complete metric space \((M, d)\) such that

\[
\sum_{n \in \mathbb{N}} d(x_n, x_{n+1})
\]

is a convergent series. Show that the sequence \( \{x_n\} \) is convergent in \( M \).
Problem 7.

Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \([0, 1]\) which converges pointwise on the interval \([0, 1]\). Suppose that \(f_n\) is continuously differentiable on \((0, 1)\) and that

\[
\int_0^1 |f_n'(x)|^2 \, dx \leq 1, \quad n = 1, 2, 3, \ldots
\]

Prove that
(a) \( \{f_n\}_{n \in \mathbb{N}} \) is equicontinuous on \([0, 1]\);
(b) \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly on \([0, 1]\).
Problem 8.

Find the maximal area of all triangles that can be inscribed in an ellipse with semiaxes $a$ and $b$. 