ERGODIC THEORY, WINTER 2011

Final Exam

Student's name:

Problem 1.

Show that there are uncountably many ergodic measures for the linear doubling map of the circle $E_2(x) = 2x \pmod 1$.

Problem 2.

Prove that the arithmetic mean of the cubes of digits appearing in the base 10 expansion of Lebesgue-a.e. $x \in [0,1)$ is well defined (and is the same for almost every x), i.e. prove that if $x = \sum_{j=0}^{\infty} \frac{x_j}{10^{j+1}},\ x_j \in \{0,1,\ldots,9\}$ then

$$\lim_{n\to\infty} \frac{1}{n}(x_0^3 + x_1^3 + x_2^3 + \ldots + x_{n-1}^3)$$

exists a.e. Find the value (for a.e. x) of this limit.

Problem 3.

Fix $\alpha \in \mathbb{R}, \alpha \notin \mathbb{Q}$, and define the map $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x,y) = (x+\alpha, x+y) \pmod 1.$$

Show that the Lebesgue measure is T-invariant and ergodic.

Problem 4.

Define the map $T : [0,1] \to [0,1]$ by $T(x) = 4x(1-x)$. Define the measure μ by

$$\mu(B) = \frac{1}{\pi}\int_B \frac{1}{\sqrt{x(1-x)}}dx.$$

a) Check that μ is a probability measure;

b) Show that T preserves μ;

c) Prove that $T : ([0,1], \mu) \to ([0,1], \mu)$ is ergodic;

d) Prove that $T : ([0,1], \mu) \to ([0,1], \mu)$ is mixing;

e) Show that $h_\mu(T) = \log 2$;

f) Show that $T : ([0,1], \mu) \to ([0,1], \mu)$ has countable Lebesgue spectrum.

<u>Problem 5.</u>

Let $\beta > 1$ denote the golden mean (i.e. $\beta^2 = \beta + 1$). Define $T : [0,1] \to [0,1]$ by $T(x) = \beta x \pmod 1$. Define the measure μ by $\mu(B) = \int_B \rho(x)dx$, where

$$\rho(x) = \begin{cases} \frac{1}{\frac{1}{\beta}+\frac{1}{\beta^3}}, & \text{on } [0, 1/\beta); \\ \frac{1}{\beta(\frac{1}{\beta}+\frac{1}{\beta^3})}, & \text{on } [1/\beta, 1]. \end{cases}$$

Prove that μ is an invariant ergodic measure, and show that $h_\mu(T) = \log \beta$.