Problem 1.

Show that there are uncountably many ergodic measures for the linear doubling map of the circle \( E_2(x) = 2x \mod 1 \).

Problem 2.

Prove that the arithmetic mean of the cubes of digits appearing in the base 10 expansion of Lebesgue-a.e. \( x \in [0,1) \) is well defined (and is the same for almost every \( x \)), i.e. prove that if \( x = \sum_{j=0}^{\infty} \frac{x_j}{10^{j+1}}, \ x_j \in \{0,1,\ldots,9\} \) then

\[
\lim_{n \to \infty} \frac{1}{n} (x_0^3 + x_1^3 + x_2^3 + \ldots + x_{n-1}^3)
\]

exists a.e. Find the value (for a.e. \( x \)) of this limit.

Problem 3.

Fix \( \alpha \in \mathbb{R}, \alpha \not\in \mathbb{Q} \), and define the map \( T : \mathbb{T}^2 \to \mathbb{T}^2 \) by

\[
T(x, y) = (x + \alpha, x + y) \mod 1.
\]

Show that the Lebesgue measure is \( T \)-invariant and ergodic.

Problem 4.

Define the map \( T : [0,1] \to [0,1] \) by \( T(x) = 4x(1-x) \). Define the measure \( \mu \) by

\[
\mu(B) = \frac{1}{\pi} \int_B \frac{1}{\sqrt{x(1-x)}} \, dx.
\]
a) Check that $\mu$ is a probability measure;
b) Show that $T$ preserves $\mu$;
c) Prove that $T : ([0, 1], \mu) \to ([0, 1], \mu)$ is ergodic;
d) Prove that $T : ([0, 1], \mu) \to ([0, 1], \mu)$ is mixing;
e) Show that $h_\mu(T) = \log 2$;
f) Show that $T : ([0, 1], \mu) \to ([0, 1], \mu)$ has countable Lebesgue spectrum.

Problem 5.

Let $\beta > 1$ denote the golden mean (i.e. $\beta^2 = \beta + 1$). Define $T : [0, 1] \to [0, 1]$ by $T(x) = \beta x \pmod{1}$. Define the measure $\mu$ by $\mu(B) = \int_B \rho(x) \, dx$, where

$$
\rho(x) = \begin{cases} 
\frac{1}{\beta + 1}, & \text{on } [0, 1/\beta); \\
\frac{1}{\beta(\frac{1}{\beta} + \frac{1}{\beta^2})}, & \text{on } [1/\beta, 1].
\end{cases}
$$

Prove that $\mu$ is an invariant ergodic measure, and show that $h_\mu(T) = \log \beta$. 