Problem 1.

Let \( f \) be a meromorphic function on \( \mathbb{C} \) which is analytic in a neighborhood of 0. Suppose that the power series of \( f \) at 0 has only real and positive coefficients, and its radius of convergence is equal to one. Prove that \( f \) has a pole at \( z = 1 \).

Problem 2.

Evaluate the integral

\[
\int_{\partial D(0,2)} (z - 1)e^{\frac{z}{z-1}} dz.
\]

Problem 3.

Evaluate the integral

\[
\int_{\infty}^{-\infty} \frac{\cos^3 x}{a^2 + x^2} dx
\]

where \( a \in \mathbb{R}, a > 0 \).

Problem 4.

Suppose that \( f \) is an entire function that has the form

\[
f(x + iy) = e^x (g(y) + ih(y)),
\]

where \( g, h \) are \( C^\infty \) real-valued functions of real argument. Suppose also that \( g(0) = 1 \) and \( h(0) = 0 \). Find the explicit form of \( f \). Is it uniquely determined by the given conditions?

Problem 5.

Let \( n \) be a positive integer. Prove that the polynomial

\[
f(x) = \sum_{i=1}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}
\]

has \( n \) distinct complex zeros, \( z_1, z_2, \ldots, z_n \), and they satisfy

\[
\sum_{i=1}^{n} z_i^{-k} = 0 \text{ for } 2 \leq k \leq n.
\]