COMPLEX ANALYSIS, HW # 7

Chapter 7, problem 65; Chapter 8, problem 3, and these problems:

Problem 1.

Suppose $u : U \to \mathbb{R}$ is a non-constant harmonic function on a connected open set. Prove that the set of points, where gradient of u vanishes, consists of isolated points.

Problem 2.

Is there a harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $\{z \mid u(z) = 0\}$ is an interval $[0, 1) \subset \mathbb{D}$? Give an example or prove that such function does not exist.

Problem 3.

Let $U = \mathbb{D} \backslash [0, 1)$. Give an explicit formula for a barriers at points $p_1, p_2, p_3 \in \partial U$, $p_1 = 0, p_2 = \frac{1}{2}, p_3 = i$.

Problem 4.

Find a bounded harmonic function $u : \mathbb{H} \to \mathbb{R}$, continuous on $\overline{\mathbb{H}}$, and such that for any $x \in \mathbb{R}$

$$u(x) = \frac{2x}{1 + x^2}.$$

Problem 5.

Consider $U = \mathbb{D} \backslash \{0\}$, and set $f : \partial U \to \mathbb{R}$,

$$f(z) = \begin{cases} 0, & \text{if } z \in \partial \mathbb{D}; \\ 1, & \text{if } z = 0. \end{cases}$$

Set $S = \{\psi : U \to \mathbb{R} \mid \psi$ is subharmonic and continuous in U, $\forall w \in \partial U \;\; \limsup_{z \to w} \psi(z) \le f(w)\}$. Define $u(z) = \sup_{\psi \in S} \psi(z)$. Prove that $u \equiv 0$.