COMPLEX ANALYSIS, HW # 3

Problem 1.

Let $f(z)$ be analytic in the disk $U = \{|z| < 1\}$, with $f(0) = f'(0) = 0$. Show that $g(z) = \sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ defines an analytic function on U. Moreover, show that the above function $g(z)$ satisfies

$$g(z) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) f(z)$$

if and only if $f(z) \equiv cz^2$.

Problem 2.

Show that $f(z) = \alpha e^z - z$ has only one zero in $U = \{|z| < 1\}$ if $|\alpha| < \frac{1}{3}$ and no zeros if $|\alpha| > 3$.

Problem 3.

Let $f(z)$ be analytic in the unit disk punched disk $U_0 = \{0 < |z| < 1\}$ such that there is a positive integer n with $|f^{(n)}(z)| \le |z|^{-n}$ for all $z \in U_0$. Show that $z = 0$ is a removable singularity for f.

Problem 4.

Find explicitly a conformal mapping of G onto the unit disk, where

$$G = \left\{ z = x + iy : |z| < 1 \text{ and } \ y > -1/\sqrt{2} \right\}.$$

Problem 5.

Find the integral

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1} dz$$

Problem 6.

Prove or disprove each of the statements:

a) If f is a function on the unit disc D such that $f(z)^2$ is analytic on D, then f itself is analytic.

b) If f is a continuously differentiable function on D, and if $f(z)^2$ is analytic on D, then f itself is analytic.

Problem 7.

If f is an entire function satisfying the estimate

$$|f(z)| \le 1 + |z|^{\sqrt{2010}} \text{ for every } z \in \mathbb{C},$$

show that f is a polynomial and determine the best upper bound for the degree of f.