Midterm Exam (sample)

Problem 1.

What is the image of the upper half-plane under a mapping of the form

\[ h(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \text{ are real, } ad - bc < 0? \]

Problem 2.

Show that

\[ 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \cdots \]

can be continued analytically to the full plane (i.e. show that its complete analytic continuation is an entire function).

Problem 3.

Suppose \( \{n_k\}_{k=1,2,...} \) is an increasing sequence of positive integers such that the infinite series \( \sum_{k=1}^{\infty} \frac{1}{n_k} \) diverges. Prove that if \( f \) is a bounded holomorphic function on \( \{\text{Re} \ z\} \) (the right-hand half-plane) having a zero at each \( n_k \), then \( f \) must be identically equal to zero.

Problem 4.

TRUE or FALSE: There exist a holomorphic function \( f \) on \( \{|z| < 1\} \) with the property that for every sequence \( \{z_n\} \) of points in the unit disk for which \( |z_n| \to 0 \) as \( n \to \infty \), the corresponding image sequence \( \{f(z_n)\}_{n=1,2,...} \) is an unbounded subset of \( \mathbb{C} \)?

Problem 5.

Prove that the product \( \prod_{k=1}^{\infty} \left( \frac{z^n}{n^n} + \exp \left( \frac{z}{2^n} \right) \right) \) converges uniformly on compact sets to an entire function.