Complex Analysis Math 220C

Final Exam (sample)

Problem 1.

Denote $f(z) = z + 1, g(z) = -\frac{1}{z+i}$.

a) Show that for any $\{n_1, m_1, n_2, m_2, \ldots, n_k, m_k\}$, $n_i, m_i \geq 0$, the composition

$$F_{(n_1,m_1,n_2,m_2,\ldots,n_k,m_k)} = f^{n_1} \circ g^{m_1} \circ \ldots f^{n_k} \circ g^{m_k}$$

has no singularities in the unit disc $\mathbb{D}$;

b) Prove that the collection of all maps $F_{(n_1,m_1,n_2,m_2,\ldots,n_k,m_k)}|_{\mathbb{D}} : \mathbb{D} \to \mathbb{C}$, $k \geq 1, n_i, m_i \geq 0$, is a normal family.

Problem 2.

Let $f(z)$ be a polynomial of degree $d \geq 2$. Prove that there exists a conformal automorphism φ of the complex plane $\mathbb{C}$ such that $\varphi \circ f \circ \varphi^{-1}$ is a polynomial of the form

$$z^d + a_2 z^{d-2} + a_3 z^{d-3} + \ldots + a_d.$$

Problem 3.

TRUE or FALSE: If $f(z)$ is an entire function that takes every complex value, then $g(z) = f(z) + z$ also takes every complex value.

Problem 4.

Let $\mathbb{D}$ be the unit disc, and suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function that has two distinct fixed points. Prove that $g(z) \equiv z$.

Problem 5.

Consider the function $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$.

a) Find the radius of convergence of the series that defines f;

b) Using the fact that $\frac{1}{\sqrt{n}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-nx^2} dx$, show that

$$f(z) = \frac{2z}{\sqrt{\pi}} \int_0^{\infty} \frac{dx}{e^{x^2} - z};$$

c) Does f allow analytic continuation to a domain larger than the disc of convergence of the series $\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$?