COMPLEX ANALYSIS MATH 220C

Midterm Exam (Sample)

Problem 1.

Here is a "counterexample" to the Little Picard Theorem. Consider the function $f(z) = e^{e^z}$. Since exponential function does not have zeros, the function f also does not have zeros. At the same time, since $e^z \neq 0$, the function $f(z) = \exp(e^z)$ does not take the value $\exp(0) = 1$. Therefore, $f(z) \notin \{0, 1\}$ for all $z \in \mathbb{C}$. What is the problem with this argument?

Problem 2.

Suppose $f : \mathbb{C} \to \bar{\mathbb{C}}$ is meromorphic doubly periodic function with periods 1 and i, i.e.

$$f(z+1) = f(z), \quad f(z+i) = f(z)$$

for all $z \in \mathbb{C}$. Let $\Pi = \{z = x + iy \mid 0 < x < 1, 0 < y < 1\}$. Suppose also that f has two simple poles in Π, and is holomorphic at al other points of Π and on $\partial\Pi$.

a) Prove that the sum of residues of f at those two poles is equal to zero;

b) Can one omit the condition that the poles are simple?

Problem 3.

Let f be a bounded analytic function in the upper half-plane $\mathbb{H}$. Suppose that $f(in) = e^{-n}$ for all $n \in \mathbb{N}$. Find $f(1+i)$. (You need to explain why the value that you found is the only possible.)

Problem 4.

Suppose f is holomorphic in the unit disk, and $f(2z) = 2f(z)f'(z)$ whenever $|z| < 1/2$. Prove that f is the restriction to the unit disk of some entire function.

Problem 5.

a) Show that there exists a holomorphic function f in a neighborhood of 0 such that $f(0) = 0$ and $f(z) = z + (1/2)f^2(z)$.

b) Show that f admits unrestricted analytic continuation in $\mathbb{C}\backslash\{1/2\}$.

c) Is there a holomorphic function g on $\mathbb{C}\backslash\{1/2\}$ that coincides with f in a neighborhood of 0? Justify your answer.