Solution of Homework 1

Problem (1.4):

Solution: By definition,

$$\begin{aligned} \text{RHS} &= |z+w|^2 \\ &= (z+w)(\overline{z+w}) \\ &= (z+w)(\overline{z}+\overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2Re(z\overline{w}) \\ &= \text{LHS} \end{aligned}$$

Similarly,

$$|z+w|^2 = (z+w)\overline{(z+w)}$$
$$|z-w|^2 = (z-w)\overline{(z-w)}$$

We can get (b) ■

Problem (1.9):

Solution: Here is one way to prove that ϕ is one to one and onto.

1: we can check that for $z \in D$,

$$\begin{aligned} \phi(z) &= i\frac{1-z}{1+z} \\ &= i\frac{(1-z)(\overline{1+z})}{(1+z)(\overline{1+z})} \\ &= i\frac{1-|z|^2+(\overline{z}-z)}{|1+z|^2} \end{aligned}$$

so,the imaginary part of $\phi(z)$ is $\frac{1-|z|^2}{|1+z|^2} > 0$, $\phi(z)$ is well defined from D to U.

2: It is obvious to show that if $\phi(z_1) = \phi(z_2)$, then $z_1 = z_2$. So ϕ is injective.

3: Choose $\varphi(w) = \frac{i-w}{i+w}$, check it in the same way as in 1, it is a well defined map from U to D, and $\varphi \circ \phi(w) = \phi \circ \varphi(w) = w$. ■

Problem (1.12)

Solution:

If we want to solve the equation like: $z^n = z_0$ for a fixed integer n, and complex number z_0. We can write

$$z_0 = |z_0|e^{i(\theta_0+2k\pi)}$$

Then $z_k = |z_0|^{\frac{1}{n}} e^{i\frac{(\theta_0+2k\pi)}{n}}$, for $k \in 0, 1, 2, ..., n-1$ is all the roots for $z^n = z_0$.

Using the result above, we have

(1)$z_1 = 2^{\frac{1}{10}} e^{i\pi\frac{1}{20}}$, $z_2 = 2^{\frac{1}{10}} e^{i\pi\frac{9}{20}}$, $z_3 = 2^{\frac{1}{10}} e^{i\pi\frac{17}{20}}$, $z_4 = 2^{\frac{1}{10}} e^{i\pi\frac{25}{20}}$, $z_5 = 2^{\frac{1}{10}} e^{i\pi\frac{33}{20}}$.

(2)$z_1 = e^{i\pi\frac{1}{2}}$, $z_2 = e^{i\pi\frac{7}{6}}$, $z_3 = e^{i\pi\frac{11}{6}}$.

(3)$z_1 = e^{i\pi\frac{1}{6}}$, $z_2 = e^{i\pi\frac{1}{2}}$, $z_3 = e^{i\pi\frac{5}{6}}$,$z_4 = e^{i\pi\frac{7}{6}}$, $z_5 = e^{i\pi\frac{3}{2}}$,$z_6 = e^{i\pi\frac{11}{6}}$.

(4)$z_1 = e^{i\pi\frac{5}{12}}$, $z_2 = e^{i\pi\frac{17}{12}}$.■

Problem (1.29)

Solution:

By definition,

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Using the formula above,we have

(1)$x + \frac{i}{2}$.

(2)$\frac{i}{2} + yi$.

(3)0.

(4)$2z-3z^2$. ■

Problem (1.43)

Solution:

f is holomorphic on U, then

$$\frac{\partial f}{\partial \overline{z}} = 0$$
$$\frac{\partial \overline{f(z)}}{\overline{z}} = \overline{\frac{\partial f(z)}{\partial z}}$$
$$\triangle \overline{f(z)} = 0$$

So we have:

$$\begin{aligned}
\triangle |f(z)|^2 &= 4\frac{\partial}{\partial z}(\frac{\partial}{\partial \overline{z}}|f(z)|^2) \\
&= 4\frac{\partial}{\partial z}(\frac{\partial}{\partial \overline{z}}f(z)\overline{f(z)}) \\
&= 4\frac{\partial}{\partial z}[\frac{\partial f(z)}{\overline{z}}\overline{f(z)} + \frac{\partial \overline{f(z)}}{\overline{z}}f(z)] \\
&= 4\frac{\partial}{\partial z}(\frac{\partial \overline{f(z)}}{\overline{z}}f(z)) \\
&= 4\frac{\partial f(z)}{\partial z}[\frac{\partial f(z)}{\partial z}\frac{\overline{f(z)}}{\partial z} + f(z)\frac{\partial}{\partial z}\frac{\overline{\partial f(z)}}{\partial z}] \\
&= 4[|\frac{\partial f(z)}{\partial z}|^2 + f(z)\triangle \overline{f(z)}] \\
&= 4|\frac{\partial f(z)}{\partial z}|^2
\end{aligned}$$

Problem (1.47)

Solution:

Suppose $f(x,y) = u(x,y) + v(x,y)$, $u(x,y)$ and $v(x,y)$ are harmonic real function. Then

$$|f(x,y)| = \sqrt{u(x,y)^2 + v(x,y)^2}$$

So,

$$\frac{\partial \log|f(x,y)|}{\partial x} = \frac{u(x,y)\frac{\partial u(x,y)}{\partial x} + v(x,y)\frac{\partial v(x,y)}{\partial x}}{u(x,y)^2 + v(x,y)^2}$$

$$\frac{\partial \log|f(x,y)|}{\partial y} = \frac{u(x,y)\frac{\partial u(x,y)}{\partial y} + v(x,y)\frac{\partial v(x,y)}{\partial y}}{u(x,y)^2 + v(x,y)^2}$$

And,

$$\begin{aligned}\frac{\partial^2 \log|f(x,y)|}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{u(x,y)\frac{\partial u(x,y)}{\partial x} + v(x,y)\frac{\partial v(x,y)}{\partial x}}{u(x,y)^2 + v(x,y)^2}\right) \\ &= \frac{u(x,y)\frac{\partial^2 u(x,y)}{\partial x^2} + v(x,y)\frac{\partial^2 v(x,y)}{\partial x^2} + (\frac{\partial u(x,y)}{\partial x})^2 + (\frac{\partial v(x,y)}{\partial x})^2}{u(x,y)^2 + v(x,y)^2} \\ &- \frac{2(u(x,y)\frac{\partial u(x,y)}{\partial x} + v(x,y)\frac{\partial v(x,y)}{\partial x})^2}{(u(x,y)^2 + v(x,y)^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \log|f(x,y)|}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{u(x,y)\frac{\partial u(x,y)}{\partial y} + v(x,y)\frac{\partial v(x,y)}{\partial y}}{u(x,y)^2 + v(x,y)^2}\right) \\ &= \frac{u(x,y)\frac{\partial^2 u(x,y)}{\partial y^2} + v(x,y)\frac{\partial^2 v(x,y)}{\partial y^2} + (\frac{\partial u(x,y)}{\partial y})^2 + (\frac{\partial v(x,y)}{\partial y})^2}{u(x,y)^2 + v(x,y)^2} \\ &- \frac{2(u(x,y)\frac{\partial u(x,y)}{\partial y} + v(x,y)\frac{\partial v(x,y)}{\partial y})^2}{(u(x,y)^2 + v(x,y)^2)^2}\end{aligned}$$

Since f is holomorphic, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So we have

$$\frac{\partial^2 \log|f|}{\partial x^2} + \frac{\partial^2 \log|f|}{\partial y^2} = 0$$

$\log|f|$ is harmonic. ■

Problem (2.4) **Solution:**

(1)

choose $\gamma(t) = e^{it}$, $t \in [0, 2\pi)$

$$\oint_{\gamma} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} d(e^{it}) = \int_0^{2\pi} \frac{1}{e^{it}} e^{it} i dt = 2i\pi$$

(2)Let us do the calculate in 4 lines,

$L1$:$z = 1 + yi, y \in [-1, 1]$,

$$\oint_{-L_1} \overline{z} + z^2 \overline{z} = -i\frac{14}{3}$$

$L2$:$z = -1 + yi, y \in [-1, 1]$,

$$\oint_{L_2} \overline{z} + z^2 \overline{z} = -i\frac{14}{3}$$

$L3$:$z = x - i, x \in [-1, 1]$,

$$\oint_{-L_3} \overline{z} + z^2 \overline{z} = i\frac{2}{3}$$

$L4$:$z = x + i, x \in [-1, 1]$,

$$\oint_{L_4} \overline{z} + z^2 \overline{z} = -i\frac{2}{3}$$

Add them up,

$$\oint_{\gamma} \overline{z} + z^2 \overline{z} = -8i$$

(3) By cauchy integral formula, since $f(z) = \frac{z}{8+z^2}$ is holomorphic in the domain with boundary γ, we know that

$$\oint_\gamma \frac{z}{8+z^2} dz = 0$$

(4) Use the similar way in(2).■