Solution of Homework 1

Problem (1.4):
Solution: By definition,

\[
\text{RHS} = |z + w|^2 = (z + w)(\bar{z} + \bar{w}) = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} = |z|^2 + |w|^2 + 2\text{Re}(z\bar{w}) = \text{LHS}
\]

Similarly,

\[
|z + w|^2 = (z + w)(\bar{z} + \bar{w})
\]

We can get (b) ■

Problem (1.9):
Solution: Here is one way to prove that \( \phi \) is one to one and onto.

1: we can check that for \( z \in D \),

\[
\phi(z) = i \frac{1 - z}{1 + z} = i \frac{(1 - z)(1 + \bar{z})}{(1 + z)(1 + \bar{z})} = i \frac{1 - |z|^2 + (\bar{z} - z)}{|1 + z|^2}
\]

so, the imaginary part of \( \phi(z) \) is \( \frac{1 - |z|^2}{|1 + z|^2} > 0 \), \( \phi(z) \) is well defined from \( D \) to \( U \).

2: It is obvious to show that if \( \phi(z_1) = \phi(z_2) \), then \( z_1 = z_2 \). So \( \phi \) is injective.
3: Choose $\varphi(w) = \frac{i-w}{i+w}$, check it in the same way as in 1, it is a well defined map from $U$ to $D$, and $\varphi \circ \phi(w) = \phi \circ \varphi(w) = w$. ■

Problem (1.12)
Solution:
If we want to solve the equation like: $z^n = z_0$ for a fixed integer $n$, and complex number $z_0$. We can write

$$z_0 = |z_0| e^{i(\theta_0 + 2k\pi)}$$

Then $z_k = |z_0| \frac{1}{n} e^{i(\theta_0 + 2k\pi)/n},$ for $k \in 0, 1, 2, ..., n - 1$ is all the roots for $z^n = z_0$.

Using the result above, we have

(1) $z_1 = 2^{\frac{1}{n}} e^{i\pi \frac{1}{n}},$ $z_2 = 2^{\frac{1}{n}} e^{i\pi \frac{9}{n}},$ $z_3 = 2^{\frac{1}{n}} e^{i\pi \frac{17}{n}},$ $z_4 = 2^{\frac{1}{n}} e^{i\pi \frac{25}{n}},$ $z_5 = 2^{\frac{1}{n}} e^{i\pi \frac{33}{n}}.$

(2) $z_1 = e^{i\pi \frac{1}{2}},$ $z_2 = e^{i\pi \frac{7}{6}},$ $z_3 = e^{i\pi \frac{11}{6}}.$

(3) $z_1 = e^{i\pi \frac{5}{12}},$ $z_2 = e^{i\pi \frac{11}{12}},$ $z_3 = e^{i\pi \frac{9}{12}},$ $z_4 = e^{i\pi \frac{5}{12}},$ $z_5 = e^{i\pi \frac{7}{12}},$ $z_6 = e^{i\pi \frac{11}{12}}.$

(4) $z_1 = e^{i\pi \frac{17}{12}},$ $z_2 = e^{i\pi \frac{11}{12}}.$ ■

Problem (1.29)
Solution:
By definition,

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$$

Using the formula above, we have

(1) $x + \frac{i}{2}.$

(2) $\frac{i}{2} + yi.$
Problem (1.43)

Solution:

If $f$ is holomorphic on $U$, then

$$\frac{\partial f}{\partial \bar{z}} = 0$$

$$\frac{\partial f(z)}{\bar{z}} = \frac{\partial f(z)}{z}$$

$$\Delta f(z) = 0$$

So we have:

$$\Delta |f(z)|^2 = 4 \frac{\partial}{\partial z} \left( \frac{\partial}{\partial \bar{z}} |f(z)|^2 \right)$$

$$= 4 \frac{\partial}{\partial z} \left( \frac{\partial}{\partial \bar{z}} f(z) \bar{f}(z) \right)$$

$$= 4 \frac{\partial}{\partial z} \left[ \frac{\partial f(z)}{\bar{z}} f(z) + \frac{\partial f(z)}{\bar{z}} f(z) \right]$$

$$= 4 \frac{\partial}{\partial z} \left( \frac{\partial f(z)}{\bar{z}} f(z) \right)$$

$$= 4 \frac{\partial f(z)}{\partial z} \frac{\partial f(z)}{\partial \bar{z}} + f(z) \frac{\partial \bar{f}(z)}{\partial z}$$

$$= 4 \left| \frac{\partial f(z)}{\partial z} \right|^2 + f(z) \Delta \bar{f}(z)$$

$$= 4 \left| \frac{\partial f(z)}{\partial z} \right|^2$$

Problem (1.47)

Solution:
Suppose \( f(x, y) = u(x, y) + v(x, y), \) \( u(x, y) \) and \( v(x, y) \) are harmonic real function. Then
\[
|f(x, y)| = \sqrt{u(x, y)^2 + v(x, y)^2}
\]
So,
\[
\frac{\partial \log |f(x, y)|}{\partial x} = \frac{u(x, y)\frac{\partial u(x, y)}{\partial x} + v(x, y)\frac{\partial v(x, y)}{\partial x}}{u(x, y)^2 + v(x, y)^2}
\]
\[
\frac{\partial \log |f(x, y)|}{\partial y} = \frac{u(x, y)\frac{\partial u(x, y)}{\partial y} + v(x, y)\frac{\partial v(x, y)}{\partial y}}{u(x, y)^2 + v(x, y)^2}
\]
And,
\[
\frac{\partial^2 \log |f(x, y)|}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{u(x, y)\frac{\partial u(x, y)}{\partial x} + v(x, y)\frac{\partial v(x, y)}{\partial x}}{u(x, y)^2 + v(x, y)^2} \right)
\]
\[
= \frac{u(x, y)\frac{\partial^2 u(x, y)}{\partial x^2} + v(x, y)\frac{\partial^2 v(x, y)}{\partial x^2} + \left( \frac{\partial u(x, y)}{\partial x} \right)^2 + \left( \frac{\partial v(x, y)}{\partial x} \right)^2}{u(x, y)^2 + v(x, y)^2}
\]
\[
- \frac{2(u(x, y)\frac{\partial u(x, y)}{\partial x} + v(x, y)\frac{\partial v(x, y)}{\partial x})^2}{(u(x, y)^2 + v(x, y)^2)^2}
\]
\[
\frac{\partial^2 \log |f(x, y)|}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{u(x, y)\frac{\partial u(x, y)}{\partial y} + v(x, y)\frac{\partial v(x, y)}{\partial y}}{u(x, y)^2 + v(x, y)^2} \right)
\]
\[
= \frac{u(x, y)\frac{\partial^2 u(x, y)}{\partial y^2} + v(x, y)\frac{\partial^2 v(x, y)}{\partial y^2} + \left( \frac{\partial u(x, y)}{\partial y} \right)^2 + \left( \frac{\partial v(x, y)}{\partial y} \right)^2}{u(x, y)^2 + v(x, y)^2}
\]
\[
- \frac{2(u(x, y)\frac{\partial u(x, y)}{\partial y} + v(x, y)\frac{\partial v(x, y)}{\partial y})^2}{(u(x, y)^2 + v(x, y)^2)^2}
\]
Since \( f \) is holomorphic, we have
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
\]
\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]
So we have
\[
\frac{\partial^2 \log |f|}{\partial x^2} + \frac{\partial^2 \log |f|}{\partial y^2} = 0
\]

\(\log |f|\) is harmonic. ■

**Problem (2.4) Solution:**

(1) choose \(\gamma(t) = e^{it}, t \in [0, 2\pi]\)

\[
\oint_{\gamma} \frac{1}{z}dz = \int_{0}^{2\pi} \frac{1}{e^{it}}d(e^{it}) = \int_{0}^{2\pi} \frac{1}{e^{it}}e^{it}idt = 2i\pi
\]

(2) Let us do the calculate in 4 lines,

**L1:** \(z = 1 + yi, y \in [-1, 1],\)

\[
\oint_{-L_1} \bar{z} + z^2\bar{z} = -i\frac{14}{3}
\]

**L2:** \(z = -1 + yi, y \in [-1, 1],\)

\[
\oint_{L_2} \bar{z} + z^2\bar{z} = -i\frac{14}{3}
\]

**L3:** \(z = x - i, x \in [-1, 1],\)

\[
\oint_{-L_3} \bar{z} + z^2\bar{z} = i\frac{2}{3}
\]

**L4:** \(z = x + i, x \in [-1, 1],\)

\[
\oint_{L_4} \bar{z} + z^2\bar{z} = -i\frac{2}{3}
\]

Add them up,

\[
\oint_{\gamma} \bar{z} + z^2\bar{z} = -8i
\]
(3) By Cauchy integral formula, since \( f(z) = \frac{z}{8+z^2} \) is holomorphic in the domain with boundary \( \gamma \), we know that

\[
\oint_{\gamma} \frac{z}{8+z^2} \, dz = 0
\]

(4) Use the similar way in (2). ■