Solution of Homework 2

Problem (2.25):

Solution:

(a) For example, $f(z) = z\overline{z} - 1 = |z|^2 - 1$ is a C^1 function. we know that $\frac{\partial f(z)}{\partial \overline{z}} = 0$ if and only if $z = 0$. So f is not holomorphic on any open set in $D(0,1)$. But

$$\oint_{\gamma} f(\varsigma)d\varsigma = \oint_{\gamma} (1-1)d\varsigma = 0$$

(b)NO. For example, $f(z) = z\overline{z} = |z|^2$ is a C^1 function but not holomorphic on $D(0,1)$. And for all $0 < r < 1$

$$\oint_{\partial D(0,r)} f(\varsigma)d\varsigma = \int_0^{2\pi} r^2 d(re^{i\theta}) = \int_0^{2\pi} r^3 ie^{i\theta} d\theta = 0$$

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Problem (2.28):

Solution:

(a) On $\partial D(8i,2)$, $z = 8i + 2e^{i\theta}$ for $0 \le \theta < 2\pi$.

$$\oint_{\partial D(8i,2)} z^3 dz = \int_0^{2\pi} (8i + 2e^{i\theta})^3 d(8i + 2e^{i\theta}) = \int_0^{2\pi} (8i + 2e^{i\theta})^3 2ie^{i\theta} d\theta = 0$$

(b) On $\partial D(6+i,3)$, $z = 6 + i + 3e^{i\theta}$ for $0 \le \theta < 2\pi$.

$$\begin{aligned}
&\oint_{\partial D(6+i,3)} z^3 dz \\
=& \int_0^{2\pi} (\overline{6+i+3e^{i\theta}} - i)^2 d(6+i+3e^{i\theta}) \\
=& \int_0^{2\pi} (6-2i+3e^{-i\theta})^2 (3ie^{i\theta}) d\theta \\
=& \int_0^{2\pi} (6-2i)^2 3ie^{i\theta} d\theta + \int_0^{2\pi} 2(6-2i)3e^{-i\theta}3ie^{i\theta} d\theta + \int_0^{2\pi} (3e^{-i\theta})^2 3ie^{i\theta} d\theta \\
=& \ 0 + 18i(6-2i)\int_0^{2\pi} d\theta + 0 \\
=& \ 72\pi(1+3i)
\end{aligned}$$

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Problem (2.29)

Solution:

(a)$f(z) = \frac{1}{z+2}$ is holomorphic in D(0,1). (That is because the only singularity of $f(z)$ is at $z = -2$, but $-2 \overline{\in} D(0,1)$) So by Cauchy integral formula,

$$\oint_{\partial D(0,1)} \frac{1}{\varsigma + 2} d\zeta = 0$$

(b) let On $\partial D(0,2)$, $z = 2e^{i\theta}$ for $0 \leq \theta < 2\pi$.

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(0,2)} \frac{1}{\varsigma + 1} d\varsigma = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{2e^{i\theta} + 1} d(2e^{i\theta}) = 1$$

The other way for this question, set $f(z) \equiv 1$ on $D(0,2)$. By Cauchy integral formula, for $w \in D(0,2)$,

$$\oint_{\partial D(0,2)} \frac{f(z)}{z-w} = f(w)$$

So:

$$f(z)=\frac{1}{2\pi i}\oint\limits_{\partial D(0,2)}\frac{1}{\varsigma+1}d\varsigma=\frac{1}{2\pi i}\oint\limits_{\partial D(0,2)}\frac{f(z)}{\varsigma-(-1)}d\varsigma=f(-1)=1$$

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Problem (11.3)

Solution: Suppose to the contrary that they are homotopic. By Theorem (Cauchy integral theorem for multiply connected domains), Since $f(z)=\frac{1}{z}$ is holomorphic on U, then

$$\oint\limits_{\gamma_1-\gamma_2}\frac{1}{z}dz=0$$

We can calculate that the integral above is not equal to 0. Contradiction.

Problem (11.5)

Solution:

Suppose there is a closed curve $\gamma\subset\mathcal{U}$, We just need to prove that γ is homotopic to a point. Because γ is compact and by the inclusion relationship of U_k's, we know that there must exist a j, such that $\gamma\subset U_j$. Moreover, U_j is topologically simply connected open set, So γ is homotopic to a point in U_j and thus homotopic to a point in (U).

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Problem (11.15) **Solution:**

For example, $f(z)=e^z, z\in\mathbb{C}$. And $f(\mathbb{C})=\mathbb{C}\setminus(0,0)$ is not topologically simply connected.

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Problem (11.21) **Solution:**

Define H(s,t) as showed in textbook, we find a continuous function $H : [0, 1] \times [0, 1] \longrightarrow U$ which is a homotopy between γ_1 and γ_3.

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