Solution of Homework 3

Problem (2.20):

Solution:

$F \in C^0(\overline{D})$, and F is holomorphic on D. choose $z \in D$, since F is holomorphic in z, so $\forall \varepsilon > 0$, there exist a $\delta > 0$, we have $|F(z) - F(\xi)| < \varepsilon$ when$|\xi - z| < \delta$. Choose $\rho < \delta$, such that $B(z, \rho) \subset D$. let $\gamma_\rho = \{\xi : |\xi - z| = \rho\}$.

Let D' be the area between γ and γ_ρ, So $\frac{F(\xi)}{\xi - z}$ is holomorphic in D', continuous in $\overline{D'}$. So we have

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{F(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int\limits_{\gamma_\rho} \frac{F(\xi)}{\xi - z} d\xi$$

And

$$\frac{1}{2\pi i}\int\limits_{\gamma_\rho} \frac{1}{\xi - z} d\xi = 1$$

$$F(z) = \frac{1}{2\pi i}\int\limits_{\gamma_\rho} \frac{F(z)}{\xi - z} d\xi$$

.

Above all,

$$\begin{aligned}
& |F(z) - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{F(\xi)}{\xi - z} d\xi| \\
= \; & |\frac{1}{2\pi i}\int\limits_{\gamma_\rho} \frac{F(z)}{\xi - z} d\xi - \frac{1}{2\pi i}\int\limits_{\gamma_\rho} \frac{F(\xi)}{\xi - z} d\xi| \\
= \; & \frac{1}{2\pi}|\int\limits_{\gamma_\rho} \frac{F(z) - F(\xi)}{\xi - z}| d\xi \\
\leq \; & \frac{1}{2\pi}\frac{\varepsilon}{\rho} 2\pi\rho \\
= \; & \varepsilon
\end{aligned}$$

let $\varepsilon \longrightarrow 0$. ■

Problem (2.21):

Solution:

Choose $f(z) = \overline{z}$. Then $F(1) = 1$.

choose $z_n = 1 - \frac{1}{n}$, And $F(z_n) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\overline{\zeta}}{\zeta - z} d\zeta \longrightarrow 0$ as $n \longrightarrow \propto$. ■

Problem (2.36)

Solution:

According to $\frac{1}{(z-1)(z-2i)} = \frac{1+2i}{5}\frac{1}{z-1} - \frac{1-2i}{5}\frac{1}{z-2i}$, So

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{1}{(\gamma-1)(\gamma-2i)} d\gamma = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{1+2i}{5}\frac{1}{\gamma-1} d\gamma - \frac{1-2i}{5}\frac{1}{\gamma-2i} d\gamma = 0$$

■

Problem (2.37)

Solution:

$$\frac{z^2+z}{(z-2i)(z+3)} = 1 - \frac{18-12i}{13}\frac{1}{z+3} - \frac{8-14i}{13}\frac{1}{z-2i}$$

, By Cauchy integra formula,

$$\begin{aligned} & \int\limits_{\gamma} \frac{\zeta^2+\zeta}{(\zeta-2i)(\zeta+3)} d\zeta \\ = & \int\limits_{\gamma} [1 - \frac{18-12i}{13}\frac{1}{\zeta+3} - \frac{8-14i}{13}\frac{1}{\zeta-2i}] d\zeta \\ = & \ 2 - 2i \end{aligned}$$

■

Problem (2.38)

Solution:

$$\frac{1}{(z-1)(z+1)} = \frac{1}{2}[\frac{1}{z-1} - \frac{1}{z+1}]$$

Use the Cauchy integra formula, it is easy to calculate

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{(\zeta-1)(\zeta+1)}d\zeta = \frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{(\zeta-1)(\zeta+1)}d\zeta$$

It does not contradict to the cauchy integral formula because $\frac{1}{(z-1)(z+1)}$ is holomorphic in $D(-1,1) \setminus \{-1\}$. $\frac{1}{(z-1)(z+1)}$ is holomorphic in $D(1,1) \setminus \{1\}$. ■

Problem (2.39)

Solution:

$F(z) = \frac{\lambda}{z}$ is holomorphic in annulus$\{z : \frac{1}{2} < z < 2\}$. And

$$\frac{1}{2\pi i}\int_{\gamma} F(z)dz = \lambda$$

. ■

Problem (3.24)

Solution:

The statement is right.

If $\sum a_j z^j$ is convergent on D(0,r). Since$\sum \varepsilon z^j = \varepsilon\frac{z}{1-z}$ for $|z| < 1$. so$\sum \varepsilon z^j$ is convergent for $|z| < 1$.

So we choose $r' = min\{r, 1\}$, then we can make a conclusion that $\sum(a_j + \varepsilon)z^j$ is convergent for some $0 < r' < r$ ■