Solution of Homework 4

Problem (3.30)

Solution:

(a)By cauchy integra formula, for arbitary $r > 0$

$$\frac{\partial^k}{\partial z^k} f(p) = \frac{k!}{2\pi i} \oint\limits_{\partial D(p,r)} \frac{f(\zeta)}{(\zeta - p)^{k+1}} d\zeta$$

Let $C_r = \sup\limits_{D(0,r)} f(z)$

$$\begin{aligned} & |\frac{\partial^k}{\partial z^k} f(P)| \\ \leq & \frac{k!}{2\pi} \oint\limits_{\partial D(p,r)} \frac{|f(\zeta)|}{(|\zeta - p|)^{k+1}} d\zeta \\ = & \frac{k!}{2\pi r^{k+1}} \oint\limits_{\partial D(p,r)} |f(\zeta)| d\zeta \\ = & \frac{k!}{2\pi r^{k+1}} M 2\pi r \\ = & \frac{k!}{r^k} C_r \end{aligned}$$

Choose $r = 1$, then we have

$$|\frac{\partial^k}{\partial z^k} f(P)| \leq C_r k!$$

(b) No.

For example, $f(z) = e^z$, then

$$|\frac{\partial^k}{\partial z^k} f(P)| = e^p$$

we can not find a polynomial $p(k)$ such that

$$|\frac{\partial^k}{\partial z^k} f(P)| \leq p(k)$$

■

Problem (3.31)

Solution:

$f^{(k)}$ is a polynomial, suppose the degree of $f^{(k)}$ is m_k, then $f^{(n)}(z) \equiv 0$ for $n > k + m_k$

Then $f(z) = \sum\limits_{k=0}^{\infty} \frac{f^k}{k!}(z-p)^k = \sum\limits_{k=0}^{n} \frac{f^k}{k!}(z-p)^k$ is a polynomial. ■

Problem (3.33)

Solution:

(a) Since f is holomorphic in D(0,r), then $f^2(z)$ is holomorphic too.

By cauchy integral formula, for $z = se^{i\theta}$,$0 < s < r$.

$$f^2(0) = \frac{1}{2\pi i} \int\limits_{\partial D(0,s)} \frac{f^2(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f^2(se^{i\theta})}{se^{i\theta}} ise^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta$$

Multiply both sides by s,

$$sf^2(0) = \frac{s}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta$$

Integrate in s from 0 to r,

$$\int_0^r sf^2(0) ds = \int_0^r \frac{s}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta ds$$

And

$$\int_0^r sf^2(0) ds = \frac{r^2}{2} f^2(0)$$

$$\int_0^r \frac{s}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta ds = \frac{1}{2\pi} \oint_{D(0,r)} f^2(x,y) dx dy$$

So we have

$$f^2(0) = \frac{1}{\pi r^2} \oint_{D(0,r)} f^2(x,y) dx dy$$

.

Which implies

$$|f(0)| \leq \frac{1}{\sqrt{\pi} r} (\oint_{D(0,r)} |f(x.y)|^2 dx dy)^{\frac{1}{2}}$$

.

(b) U be an open set and K be a compact subset of U. Se can find an open cover $D(z, r_z)$ of K for $z \in K$.

Since K is compact, so there is a finite subcover. So there exists $\delta = dist(K, U) > 0$ such that $r_z > \delta$.

By the formula in (a), for any $z \in K$:

$$|f(z)| \leq \frac{1}{\sqrt{\pi} \delta} (\oint_{U} |f(x.y)|^2 dx dy)^{\frac{1}{2}}$$

So

$$\sup_K |f(z)| \leq \frac{1}{\sqrt{\pi} \delta} (\oint_{U} |f(x.y)|^2 dx dy)^{\frac{1}{2}}$$

■

Problem (3.37)

Solution:

By Cauchy estimate $|f' - f_j'| \leq \frac{|f(z) - f_j(z)|}{r}$, we know that if f_j is a sequence of holomorphic functions which converges uniformly on any compact subset to a holomorphic function f, then $f_j'(z)$ converges to $f'(z)$.

So, we know that if $f_j^{(n)} \to f^{(n)}$ uniformly on any compact subset of domain.

Moreover, $f^{(n)}$ is holomorphic therefore if it is a zero on a compact subset which does not consist of finitely many points then it has an accumulation point, hence $f^{(n)} \equiv 0$ on all subsets of D by uniqueness.

Since $\{P_k\}$ be a family of holomorphic polynomials of degree n that converge to f uniformly on compact subset of . Since f is holomorphic in it has a power series expansion around some point $p \in C$

$$f(z) = \sum_{k=0}^{\infty} \frac{\partial^k f(p)}{k!}(z-P)^k$$

However, $P_k^{n+1} \equiv 0$,therefore $f^{n+1} \equiv 0$ on any compact set hence it is true on all of . Therefore,

$$f(z) = \sum_{k=0}^{n} \frac{\partial^k f(p)}{k!}(z-P)^k$$

. So f is a polynomial of degree less than or equal to n. ■

Problem (3.39)

Solution:

Since φ is holomorphic in D(0,1), so φ_j are all bounded. By Montel's theorem, there is a subsequence of φ_j that converges to a holomorphic function f.

And we know that φ_j converges uniformly on compact sets to φ, So φ is holomorphic in D(0,1). And

$$limit_{n\to\infty}\varphi(\varphi_{n+1}(z)) = \varphi(limit_{n\to\infty}\varphi_n(z)) = \varphi(f(z)) = f(z)$$

Compose with f^{-1} on both sides to get $\varphi(z) = z$. ■

Problem (3.42)

Solution:

Suppose f has infinite many zeros in D(p,r), let $A = \{z \in D(p,r) : f(z) = 0\}$, then we can find a convergent sequence $\{x_n\} \subseteq A$, such that $x_n \to x$.

Since $f(x_n) = 0$ for any n and f is holomorphic, so $limit_{n\to\infty} f(x_n) = f(limit_{n\to\infty} x_n) = f(x) = 0$. So $f(z) \equiv 0$. we get a contradiction. Hence f can only have finite zeros in $D(p, r)$. ■

Problem (4.5) Solution:

(a) $z = 0$ is a pole of $f(z) = \frac{1}{z}$, since $\text{limit}_{z\to 0} \frac{1}{z} = \infty$.

(b) $z = 0$ is a essential singularity of $f(z) = \sin\frac{1}{z}$. Since

$$\text{limit}_{n\to\infty} \sin\frac{1}{2n\pi} = 0$$

$$\text{limit}_{n\to\infty} \sin\frac{1}{2n\pi + \frac{2}{\pi}} = 1$$

so $\text{limit}_{z\to 0} \sin\frac{1}{z}$ does not exist.

(c)$z = 0$ is a pole of $f(z) = \frac{1}{z^3} - \cos z$, since $\text{limit}_{z\to 0}(\frac{1}{z^3} - \cos z) = \infty$.

(d) $z = 0$ is a essential singularity of $f(z) = z \cdot e^{\frac{1}{z}} \cdot e^{-\frac{1}{z^2}}$, since $\text{limit}_{z\to 0}(z \cdot e^{\frac{1}{z}} \cdot e^{-\frac{1}{z^2}})$ does not exist.

(e) $z = 0$ is a removable singularity of $f(z) = \frac{\sin z}{z}$, since $\text{limit}_{z\to 0}\frac{\sin z}{z} = 1$.

(f) $z = 0$ is a pole of $f(z) = \frac{\cos z}{z}$, since $\text{limit}_{z\to 0}\frac{\cos z}{z} = \infty$.

(g) $z = 0$ is a pole of $f(z) = \frac{\sum\limits_{k=2}^{\infty} 2^k z^k}{z^3}$, since $\text{limit}_{z\to 0}\frac{\sum\limits_{k=2}^{\infty} 2^k z^k}{z^3} = \text{limit}_{z\to 0}\frac{1}{z^3}\frac{(2z)^2}{1-2z} = \text{limit}_{z\to 0}\frac{4}{z(1-z)} = \infty$ ■