Solution of Homework 5

Problem (4.3):

Solution:

(a) Iff has an essential singularity at P, then $\frac{1}{f}$ has a essential singularity at P.

(b) Iff has an pole at P, then $\frac{1}{f}$ has a removable singularity at P.

(c) Iff has an removable singularity at P, and $lim_{z\to p}f(z) = 0$, then $\frac{1}{f}$ has a pole at p. Iff has an pole at P, and $lim_{z\to p}f(z) \neq 0$,then $\frac{1}{f}$ has a removable singularity at P. ■

Problem (4.4):

Solution:

A_1 is closed under $+,\times$. But it is not closed under division. For example, $f_1(z) = 1$, $f_2(z) = z$, then 0 is removable singularity of both f_1 and f_2. But $z = 0$ is a pole of $\frac{f_1}{f_2} = \frac{1}{z}$.

A_2 is closed under $\times$. But it is not closed under addition and division. For example, $f_1(z) = \frac{1}{z}$, $f_2(z) = -\frac{1}{z}$, then 0 is pole of both f_1 and f_2. But $z = 0$ is a removable singularity of $f_1(z) + f_2(z) = 0$ and $\frac{f_1}{f_2} = \frac{\frac{1}{z}}{\frac{-1}{z}} = -1$.

A_3 is not closed under addition, multiplication and division. For example, $f_1(z) = e^{\frac{1}{z}}$, $f_2(z) = e^{\frac{-1}{z}}$, then 0 is an essential singularity of both f_1 and f_2. But $z = 0$ is a removable singularity of $f_1(z) \times f_2(z) = 1$. And if $f_1(z) = e^{\frac{1}{z}}$, $f_2(z) = -e^{\frac{1}{z}}$, then 0 is an essential singularity of both f_1 and f_2. But $z = 0$ is a removable singularity of$f_1(z) + f_2(z) = 0$ and $\frac{f_1}{f_2} = -1$.
■

Problem (4.12)

Solution:

For example, $\sum_{n=0}^{\infty} z^n$ is convergent in an annular region $\frac{1}{2} < |z| < 1$, and it is convergent for $\{z : |z| = \frac{1}{2}\}$, but divergent on $\{z : |z| = 1$. So it is an example for that a laurent series can convergent include some of the boundary.

For example, $\sum_{n=0}^{\infty} z^n$ is convergent in an annular region $\frac{1}{2} < |z| < \frac{3}{4}$, and it is convergent for both $\{z : |z| = \frac{1}{2}\}$ and $\{z : |z| = \frac{3}{4}\}$. So it is an example for that a laurent series can convergent include all of the boundary.

For example, $f(z) = \frac{1}{z-\frac{1}{z}} + \frac{1}{z-2}$ is convergent in an annular region $\frac{1}{2} < |z| < |2|$, and it is not convergent for both $\{z : |z| = \frac{1}{2}\}$ and $\{z : |z| = 2\}$. So it is an example for that a laurent series can convergent include none of the boundary. ■

Problem (4.21)

Solution: We know that P is a nonremovable singularity of $f(z)$, then it is either a pole or essential singularity.

Suppose P is a pole of $f(z)$ with order k , then

$$f(z) = \frac{a_{-k}}{(z-P)^k} + \frac{a_{-k+1}}{(z-P)^{k-1}} + \cdots + \frac{a_{-1}}{(z-P)^-1} + \sum_{n=0}^{\infty} a_n z^n$$

So we have $e^{f(z)} = \sum_{m=0}^{\infty} \frac{f(z)^m}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!} (\sum_{i=-k}^{-1} \frac{a_{-i}}{(z-P)^i} + \sum_{j=0}^{\infty} a_j z^j)^m$

we can see that there are infinite many nonzero coefficient of $e^{f(z)}$ with negative power.

Suppose P is an essential singularity of $f(z)$, then for any $0 < r < \delta$, $f(D(P,r) \setminus P)$ is dense in $\mathbb{C}$. So $\forall z \in \mathbb{C}$, there exist $x \in D(f(z), r)$, such that such that $|z - x| < \varepsilon$. Here $x = f(w), w \in D(P,r) \setminus P$. And since e^z is continuous function. then for $\forall \varepsilon_2 > 0$, there exist a $\delta > 0$ such that $|e^{f(x)} - e^z| < \varepsilon_2$. So we prove that then for any $0 < r < \delta$, $e^{f(D(P,r) \setminus P)}$ is dense in $\mathbb{C}$. Then P is an essential singularity of $e^{f(z)}$. ■

Problem (4.33)

Solution:

(a) $2i$ is a pole of $f(z) = \frac{z^2}{(z-2i)(z+3)}$ with order 1, so

$$Res_f(2i) = f(z)(z-2i)|_{z=2i} = -\frac{12}{13} + \frac{8}{13}i$$

(b) -3 is a pole of $f(z) = \frac{z^2+1}{z(z+3)^2}$ with order 2, so

$$Res_f(-3) = \frac{1}{(2-1)!}[f(z)(z+3)^2]'|_{z=-3} = \frac{8}{9}$$

(c) $i+1$ is a pole of $f(z) = \frac{e^z}{(z-i-1)^3}$ with order 3, so

$$Res_f(i+1) = \frac{1}{(3-1)!}[f(z)(z-i-1)^3]^{(2)}|_{z=i+1} = \frac{1}{2}e^{i+1}$$

(d) 2 is a pole of $f(z) = \frac{z}{(z+1)(z-2)}$ with order 1, so

$$Res_f(2) = f(z)(z-2)|_{z=2} = \frac{2}{3}$$

(e) $-i$ is a pole of $f(z) = \frac{cotz}{z^2(z+i)}$ with order 2, so

$$Res_f(-i) = \frac{1}{(2-1)!}[f(z)(z+i)^2]'|_{z=-i} = 2icot(-i) - csc^2(-i)$$

(f) 0 is a pole of $f(z) = \frac{cotz}{z(z+1)}$ with order 2, so

$$Res_f(0) = \frac{1}{(2-1)!}[f(z)z^2]'|_{z=0} = -1$$

(g) 0 is a pole of $f(z) = \frac{sinz}{z^3(z-2)(z+1)}$ with order 2, so

$$Res_f(0) = \frac{1}{(2-1)!}[f(z)z^2]'|_{z=0} = \frac{1}{4}$$

(h) π is a pole of $f(z) = \frac{cot(z)}{z^2(z+1)}$ with order 1, so

$$Res_f(\pi) = f(z)(z-\pi)|_{z=\pi} = \frac{1}{\pi^2(\pi+1)}$$

■

Problem (4.34)

Solution:

(a)

$$1$$

(b)

$$1-\frac{e^{-1}}{sin1}-\frac{e^{-\pi}}{-\pi+1}-\frac{e^{\pi}}{\pi+1}$$

(c)

$$\frac{1}{28}-\frac{e^2+e^{-2}}{16(e^2-e^{-2})}$$

(d)

$$e^{-1}-\frac{1}{2}e^{-2}-\frac{1}{2}$$

(e)

$$\frac{2e^{-3}}{(-3+3i)^3}-\frac{e^{-4}}{(-4+3i)^2}$$

(f)

$$1+2i$$

(g)

$$\frac{e^{-2}}{4}$$

(h)

$$-1+i$$

(i)

$$0$$

■

Problem (4.40) **Solution:**

By $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, we know that: $e^{z+\frac{1}{z}} = (\sum_{i=0}^{\infty} \frac{z^i}{i!})(\sum_{j=0}^{\infty} \frac{(\frac{1}{z})^j}{j!}) = \sum_{n=0}^{\infty} a_n z^n$

We know that $Res_f(0) = a_{-1}$ So we consider the coefficient of $n = i - j = -1$, $Res_f(0) = a_{-1} = \sum_{i-j=-1}^{\infty} \frac{1}{i!j!} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}$ ■