Solution of Homework 7

Problem (4.48):
Solution:
\[ \int_{0}^{+\infty} \frac{x^{\frac{1}{4}}}{1 + x^3} \, dx = \frac{\pi}{3 \sin(\frac{5\pi}{12})} \]

Problem (4.50):
\[ \int_{0}^{+\infty} \frac{1}{1 + x^3} \, dx = \frac{\pi}{3 \sin(\frac{\pi}{3})} = \frac{2\sqrt{3}\pi}{9} \]
Solution:

Problem (4.54)
Solution:
\[ \int_{-\infty}^{0} \frac{x^{\frac{1}{3}}}{1 + x^5} \, dx = \frac{\pi}{5 \sin(\frac{4\pi}{15})} \]

Problem (4.56)
Solution:
\[ \int_{-\infty}^{+\infty} \frac{x^4}{1 + x^{10}} \, dx = \frac{\pi}{5} \]
Problem (5.2)
Solution:
suppose \( f \) has zeros at \( P_1, P_2, \ldots, P_k \) with order \( \lambda_1, \lambda_2, \ldots, \lambda_k \).
then
\[
\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} \cdot g(z) dz = \sum_{i=1}^{k} \text{Res}(\frac{f'(z)}{f(z)} \cdot g(z), P_i) = \sum_{i=1}^{k} \lambda_i g(P_i)
\]

Problem (5.5)
Solution:
For example, let \( f_j(z) = (z - \frac{1}{2})(z - 1 + \frac{1}{j})^{k-\iota} \), then \( f_j(z) \) goes to \( f(z) = (z - \frac{1}{2})(z - 1)^{k-\iota} \) as \( j \to \infty \). \( f_j(z) \) has \( k \) roots in \( D(0,1) \), but \( f(z) \) only has exactly \( \iota \) roots in \( D(0,1) \).
We need to assume that \( f \) has no zeros on \( \partial D(0,1) \), then we can guarantee that \( f \) does have at least \( k \) roots.

Problem (5.8) Solution:
Since \( f(z) \neq 0 \) on \( \partial D(P,r) \) and \( \partial D(P,r) \) is compact, \( |f(z)| \geq \epsilon \) on \( \partial D(P,r) \) for some \( \epsilon > 0 \). Suppose \( |f(z) - g(z)| < \epsilon \) for all \( z \in \partial D(P,r) \), then
\[
|f(z) - g(z)| < |f(z)| + |g(z)|
\]
By Rouche’s theorem, \( f \) and \( g \) have the same number of zeros in \( D(P,r) \) counting multiplicity.