Problem 1.

Find the largest disk centered at 1 in which the Taylor series for

\[
\frac{1}{1+z^2} = \sum a_k(z - 1)^k
\]

will converge.

Problem 2.

Let \( f(z) \) be entire holomorphic function on \( \mathbb{C} \) such that \( |f(z)| \leq |\cos z| \). Prove \( f(z) = c \cos z \) for some constant \( c \).

Problem 3.

Prove that there is no entire analytic function such that

\[
\bigcup_{n=0}^{\infty} \left\{ z \in \mathbb{C} : f^{(n)}(z) = 0 \right\} = \mathbb{R}.
\]

Problem 4.

Is there an entire function \( f \) such that

\[
f \left( \frac{1}{n} \right) = f \left( -\frac{1}{n} \right) = \frac{1}{n^3}
\]

for all \( n \in \mathbb{N} \)? Justify your answer.

Problem 5.

Find the radius of convergence \( R_1 \) of the series

\[
\sum_{n=1}^{+\infty} \frac{z^n}{n^2}
\]

and show the series converges uniformly on \( \overline{D(0, R_1)} \). What is the radius of convergence \( R_2 \) of the derivative of this series? Does it converge uniformly on \( \overline{D(0, R_2)} \)?