Solution of Homework 1

Problem (6.6):
Solution:
The domain $\Omega$ is conformally equivalent to $D(0,1)\setminus\{0\}$. $f(z) = 1/z$ is a conformal map from $D(0,1)\setminus\{0\}$ to $\Omega$.
Suppose $g$ is a conformal mapping from $\Omega$ to $\Omega$. Then $f^{-1} \circ g \circ f$ is a biholomorphic-self map of $D(0,1)\setminus\{0\}$.
And
$$Aut(D(0,1)\setminus\{0\}) = \{e^{i\psi}z | \psi \in \mathbb{R}\}$$
So we know that $g = e^{-i\psi}z$.
Above all:
$$Aut(\Omega) = \{e^{i\varphi}z | \varphi \in \mathbb{R}\}$$

Problem (6.11):
Solution:
Observe that $\phi_1 \circ \phi_2^{-1} : D \to D$ is a conformal map. According to
$$Aut(D(0,1)) = \{e^{i\theta} \frac{a-z}{1-\bar{a}z} | a \in D(0,1), \theta \in [0,2\pi)\}$$
So there exist $a \in [0,2\pi), \varphi \in Aut(D(0,1))$ such that $\phi_1 = \varphi \circ \phi_2$. Then we know $\phi_1(z) = e^{i\theta} \frac{a-\phi_2(z)}{1-\bar{a}\phi_2(z)}$

Problem (6.17)
Solution:
Since $\phi(P) = P, \phi(P) = 1$, from the hint in the book, we know that $\phi(z) = P + (z-P) + h(z)$, where $h(z)$ contains the higher order terms.
Suppose $\phi(z) \neq z$. then $h(z)$ has an nonzero term, say $a_n(z - P)^n$. Since $\phi$ maps $\Omega$ to itself, so $|\phi(z)| \leq M$, where $M = \sup\{|z|, z \in \Omega\}$. By cauchy estimate, $|a_n| \leq \frac{M}{r^n}$.

And by induction $\phi^j(z) = \phi \circ \phi \circ \cdots \phi = P + (z - P) + ja_n(z - P)^n + \cdots$. But $\phi^j$ maps $\Omega$ to itself. So the same cauchy estimate gives $|ja_n| \leq \frac{M}{r^n}$ for all $j \in N$, which is impossible for nonzero $a_n$. Therefore the taylor series for $\phi$ has no next smallest nonzero term, that is $\phi(z) = P + (z - P) = z$.

Problem (6.20)
Solution:
Let $K$ be any compact subset of $\mathbb{D}$ then there exists an $r$ such that the union $E$ of all the disks $D(z, r)$ for all $z \in K$, is a compact subset of $\mathbb{D}$. We now apply the Cauchy estimate to $f - f_\alpha$ to get

$$|f'(z) - f'_\alpha(z)| \leq \frac{\sup_{z \in K}|f(z) - f_\alpha(z)|}{r}$$

Since $f_\alpha \to f$ converges uniformly, we conclude that $f'_\alpha \to f'$ uniformly. Since $K$ was an arbitrary compact set, then we know that $f'_\alpha$ is a normal family.

Problem (6.21)
Solution:
put $b = \frac{a}{|a|}$, $\phi(z) = \frac{z - b}{z + b}$ and $R(z) = \phi \circ L \circ \phi^{-1}$. Note that

$$L(b) = \frac{\frac{a}{|a|} - a}{1 - \frac{a}{|a|}} = \frac{a}{|a|} = b$$

$$L(b) = \frac{-\frac{a}{|a|} - a}{1 - \frac{a}{|a|}} = -\frac{a}{|a|} = -b$$
So
\[ R(0) = \phi \circ L \circ \phi^{-1}(0) = \phi \circ L(b) = \phi(b) = 0 \]
and
\[ R(\infty) = \phi \circ L \circ \phi^{-1}(\infty) = \phi \circ L(-b) = \phi(-b) = \infty \]

Since \( R \) is a linear fractional transformation that fixed 0 and \( \infty \). \( R(z) = \alpha z \) for some non-zero \( \alpha \in \mathbb{C} \). To determine \( \alpha \), we see that

\[ \alpha = R(1) = \phi \circ L \circ \phi^{-1}(1) = \phi \circ L(\infty) = \phi\left(\frac{-1}{a}\right) = \frac{-1 - a}{a + \frac{a}{|a|}} = \frac{-1}{1 + |a|} \]

So we know that \(|\alpha| > 1\). Now note that \( L_n(z) = \phi^{-1} \circ R^n \circ \phi(z) = \phi^{-1}(\alpha^n \phi(z)) \). As \( n \to \infty \), \( L_n(z) \to \phi^{-1}(\infty) = 1 \). By Montel’s theorem, some subsequence of \( L_j \) converges uniformly on compact sets, so does the whole sequence. \( \blacksquare \)

**Problem (6.23)**

**Solution:**

By Montel’s theorem, every subsequence of \( f_n \) has a subsequence that converges normally on compact subsets. Let \( K \subset \Omega \) be compact and suppose \( f_{n_j} \to f \) and \( f_{n_k} \to g \). Since \( \lim_{n \to \infty} f_n(z_k) \) exists for each \( k \), \( f \) and \( g \) agree on \( z_k \), so by the identity theorem \( f \equiv g \) on \( \Omega \). So all subsequences converge to the same limit, therefore \( f_n \) converges on \( K \).

\( \blacksquare \)

**Problem (6.32) Solution:**

First, \( \phi_1(z) = -i \frac{z + i}{z - i} \) maps the unit disc to the upper half plane. Secondly, \( \phi_2(z) = -z \) maps the upper half plane to the lower half plane. Then \( \phi_3 = e^{-i\alpha}z + 4 \) with \( \alpha = \arctan \frac{1}{2} \) maps the lower half plane to the area below the line \( x + 2y = 4 \).

So the linear fractional transformation is \( \phi_3 \circ \phi_2 \circ \phi_1(z) \)

\( \blacksquare \)