Solution of Homework 4

Problem (7.24):
Solution: Observe that if $u$ is harmonic function, then

$$L(u) = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} + c \frac{\partial^2 u}{\partial x \partial y} = (a - b) \frac{\partial^2 u}{\partial x^2} + c \frac{\partial^2 u}{\partial x \partial y}$$

In particular, since $\rho_\theta$ is a conformal mapping, $u \circ \rho_\theta$ is a harmonic function and

$$L(u \circ \rho_\theta) = (a - b) \frac{\partial^2 (u \circ \rho_\theta)}{\partial x^2} + c \frac{\partial^2 (u \circ \rho_\theta)}{\partial x \partial y}$$

Because $L$ commutes with rotations, we have

$$(a - b) \frac{\partial^2 (u \circ \rho_\theta)}{\partial x^2} + c \frac{\partial^2 (u \circ \rho_\theta)}{\partial x \partial y} = (a - b) \frac{\partial^2 u}{\partial x^2} \circ \rho_\theta + c \frac{\partial^2 u}{\partial x \partial y} \circ \rho_\theta$$

Thus we have the following equality:

$$(a - b)(\frac{\partial^2 (u \circ \rho_\theta)}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} \circ \rho_\theta) = c(\frac{\partial^2 u}{\partial x \partial y} \circ \rho_\theta - \frac{\partial^2 (u \circ \rho_\theta)}{\partial x \partial y})$$

now we can apply these results to the harmonic function $u(x, y) = xy$.

$$u(e^{i\theta}z) = (x \cos \theta - iy \sin \theta)(y \cos \theta + ix \sin \theta)$$
$$= x^2 \sin \theta \cos \theta + xy(cos^2 \theta - sin^2 \theta) - y^2 \sin \theta \cos \theta$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = 0$$
$$\frac{\partial^2 u}{\partial x \partial y} = 1$$
$$\frac{\partial^2 u(e^{i\theta}z)}{\partial x^2} = \sin(2\theta)$$
\[
\frac{\partial^2 u(e^{i\theta} z)}{\partial x \partial y} = \cos(2\theta)
\]

Therefore,
\[-(a - b) \sin(2\theta) = c(\cos(2\theta) - 1)\]

Then we know that \(a = b, \ c = 0\). \(\blacksquare\)

**Problem (7.25):**

**Solution:**

First, by Thm 6.3.6 \(f(z) = \frac{z - i}{z + i}\) maps \(U\) to \(D(0, 1)\) conformally.

Let \(u\) be a harmonic function from \(\overline{U}\) to \(\mathbb{R}\). Since \(U(u) \in \mathbb{R}\), thus \(u \circ f^{-1}\) is a harmonic function on a neighborhood of \(D(0, 1)\). Then for any point \(a \in D(0, 1)\), by poisson integral formula, we have

\[
u \circ f^{-1}(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \left| \frac{a}{a - e^{i\psi}} \right|^2 \right) d\psi
\]

Suppose \(f^{-1}(a) = b \in U\), then

\[
u(b) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \left| \frac{f(b)}{f(b) - e^{i\psi}} \right|^2 \right) d\psi
\]

And we know that \(f^{-1}(z) = i \frac{1 + z}{1 - z}\), thus

\[
u(b) = \frac{1}{2\pi} \int_0^{2\pi} u \left(\frac{1 + e^{i\psi}}{1 - e^{i\psi}} \right) \left(1 - \left| \frac{b - i}{b + i} \right|^2 \right) d\psi
\]

\(\blacksquare\)

**Problem (7.26):**

**Solution:**

First we know that \(\phi_1 = \frac{1 + z}{1 - z}\) is a conformal mapping with the upper semi disk and the first quadrant of the complex plane. Next, the map \(\phi_2 = z^2\) is...
a conformal map of the first quadrant with the upper half plane. Finally, the map \( \phi_3 = \frac{z-i}{z+1} \) is a conformal mapping of the upper half plane with the unit disc. Composing these maps, we get the conformal map

\[
 f(z) = \phi_3 \circ \phi_2 \circ \phi_1(z) = \frac{(1+z)^2 - i}{(1+z)^2 + i}
\]

Let \( u \) be a harmonic function from \( \overline{U} \) to \( \mathbb{R} \). Since \( U(u) \in \mathbb{R} \), Thus \( u \circ f^{-1} \) is a harmonic function on a neighborhood of \( D(0, 1) \). Then for any point \( a \in D(0, 1) \), by poisson integral formula, we have

\[
 u \circ f^{-1}(a) = \frac{1}{2\pi} \int_0^{2\pi} u \circ f^{-1}(e^{i\psi}) \frac{1 - |a|^2}{|a - e^{i\psi}|} d\psi
\]

suppose \( f^{-1}(a) = b \in U \), then

\[
 u(b) = \frac{1}{2\pi} \int_0^{2\pi} u \circ f^{-1}(e^{i\psi}) \frac{1 - |f(b)|^2}{|f(b) - e^{i\psi}|} d\psi
\]

And we know that

\[
 f^{-1}(z) = \frac{\sqrt{i(1+z)} + 1}{\sqrt{i(1+z)} - 1}
\]

, thus

\[
 u(b) = \frac{1}{2\pi} \int_0^{2\pi} u\left(\frac{\sqrt{i(1+e^{i\psi})}}{1-e^{i\psi}} + 1\right) \frac{1 - \left| \frac{(1+ib)^2 - i}{(1+ib)^2 + i - e^{i\psi}} \right|^2}{\left| \frac{(1+ib)^2 - i}{(1+ib)^2 + i - e^{i\psi}} \right|} d\psi
\]

\[\blacksquare\]

**Problem (7.32)**

**Solution:**

we know that

\[
 \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{(e^{i\theta} + z) \times (e^{i\theta} - z)}{(e^{i\theta} - z) \times (e^{i\theta} - z)} = \frac{1 - |z|^2 - e^{i\theta} \bar{z} + ze^{-i\theta}}{|e^{i\theta} - z|^2}
\]
So the real part of $\frac{e^{i\theta} + z}{e^{i\theta} - z}$ is $\frac{1 - |z|^2}{|e^{i\theta} - z|^2}$

Thus by the Poisson integral formula,

$$
\text{Re } h(z) = \text{Re } \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(e^{i\theta}) d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \text{Re } \frac{e^{i\theta} + z}{e^{i\theta} - z} u(e^{i\theta}) d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta
$$

$$
= u(z)
$$

\[ \Box \]

**Problem (7.33)**

**Solution:**

Since $u$ is harmonic on $U \{ P \}$, it has the mean value property. Let $r > 0$ be such that $D(P, 2r) \subseteq U$. Thus for $|z - P| < r$, $D(P, r) \subseteq U$.

Thus for $|z - P| < r$,

$$
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta
$$

Then by $u$ is continuous on $U$

$$
u(P) = \lim_{z \to P} u(z)
$$

$$
= \lim_{z \to P} \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \lim_{z \to P} u(z + re^{i\theta}) d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta}) d\theta
$$
So $u$ satisfies the mean value theorem on $P$, by Thm 7.4.2, it is harmonic on $U$.

Problem (7.39)  
Solution:
Suppose $U^+ = \{\text{Image part of } z > 0 | z \in \mathbb{C}\}$, $U^- = \{\text{Image part of } z < 0 | z \in \mathbb{C}\}$. Since Image part of $h(z)$ is equal to 0 for $z \in \mathbb{R}$. So by schwarz reflection principle, $h(z) = \overline{h(z)}$ for $z \in U^-$.  
For $a \in \mathbb{R}$, there exist a $b \in \mathbb{R}$ such that $h(ai) = bi$. And we know that $h(-ai) = \overline{h(ai)} = \overline{bi} = -bi$, So we get $h(ai) = -h(-ai)$. Since $\{\text{Re } z = 0\}$ has an accumulation point, so we get $h(z) = -h(z)$ by uniqueness theorem.

Problem (7.61) Solution:
suppose $u_1 = \text{Re } H$, $u_2 = \text{Re } G$. Since $u_1$ and $u_2$ are continuous on the boundary and harmonic in the disc. We get $u_1 \equiv u_2$ in $D$ by $u_1 = u_2$ on the boundary.  
And we know that $G - H$ is holomorphic function in $D$ and the real part is 0. Then we know that the image part of $G - H$ is constant.