Solution of Homework 5

Problem (7.40):
Solution:
Yes. If \( u_1 \geq u_2 \geq \cdots \) is a decreasing sequence of harmonic functions on a connected open set \( U \subset \mathbb{C} \), then either \( u_j \to \infty \) uniformly on compact sets or there is a harmonic function \( u \) on \( U \) such that \( u_j \to u \) uniformly on compact sets.

Proof: Put \( v_j = -u_j \). Apply the Harnack’s principle.

Problem (7.41):
Solution:
Let \( P = (a, b) \in U \), write out the two variable Taylor expansion of \( f \) at \( P \).

\[
 f(x, y) = f(P) + (x - a)f_x(P) + (y - b)f_y(P) + \frac{1}{2}[(x - a)^2 f_{xx}(P) + 2(x - a)(y - b)f_{xy}(P) + (y - b)^2 f_{yy}(P)] + \text{higher order terms}
\]

Let \( x = a + r \cos \theta, y = b + r \sin \theta \), then we get

\[
 f(P + re^{i\theta}) = f(P) + r \cos \theta f_x(P) + r \sin \theta f_y(P) + \frac{r^2}{2}[(\cos \theta)^2 f_{xx}(P) + 2(\cos \theta)(\sin \theta)f_{xy}(P) + (\sin \theta)^2 f_{yy}(P)] + \text{higher order terms}
\]

\( U \) is open, so there exists \( 0 < r < 1 \), s.t \( \overline{D(P, r)} \in U \). Thus since \( f \) is subharmonic,

\[
 f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})d\theta
\]

Put the Taylor expansion above in the subharmonic inequation above, the \( \sin \theta, \cos \theta \) and \( \cos 2\theta \) vanish. Then we have

\[
 f(P) \leq f(P) + 0 + 0 + \frac{r^2}{4}[f_{xx}(P) + f_{yy}(P)] = \frac{r^2}{4}\Delta f(P)
\]
So we get, $\Delta f \geq 0$ on $U$.

Problem (7.42)
Solution:
Yes. Let $P \in U$ and $K \subset U$ be a compact set containing $P$. Then there exists $r > 0$ such that $D(P, r) \subset K$. If $f = \lim f_j$, then

$$f(P) = \lim_{j \to \infty} f_j(P)$$

$$\leq \lim_{j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f_j(P + re^{i\theta})d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \lim_{j \to \infty} f_j(P + re^{i\theta})d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})d\theta$$

So $f$ is subharmonic.

Problem (7.48)
Solution:
For every point $P \in U$, there is a $D(P, r) \subset U$. Let $\varphi : \overline{D(0, 1)} \to U$ be defined by $\varphi(z) = rz + P$. Then $\varphi$ is holomorphic from $D(0, 1)$ to
so we have

\[ f(P) = f \circ \varphi(0) \leq \frac{1}{2\pi} \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \, d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \lim_{j \to \infty} f_j(P + re^{i\theta}) \, d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) \, d\theta \]

So that \( f \) is subharmonic.

\[ \square \]

**Problem (7.49)**

**Solution:**

For example \( f(x, y) = -x^2 \), then \( f^2(x, y) = x^4 \). Since \( \Delta f = -2 < 0 \), then \( f \) is not subharmonic.

And \( \Delta f^2 = 12x^2 \geq 0 \), so that \( f^2 \) is subharmonic.

\[ \square \]

**Problem (7.69)**

**Solution:**

(a)

Let \( \overline{D(P, r)} \subset U \), and let \( h \) be harmonic on a neighborhood of \( \overline{D(P, r)} \), such that \( f \leq h \) on \( \partial D(P, r) \).

Since \( h \) is harmonic, \( \Delta h = 0 \). Thus for any \( z \in D(P, r) \), \( \Delta (f - h) = \Delta f - \Delta h = \Delta f > 0 \). So \( f - h \) can not have a local maximum. However, \( f - h \) has a maximum on \( \overline{D(P, r)} \). Thus the maximum occurs on \( \partial D(P, r) \).

Then we know that \( \exists w \in \partial D(P, r) \), s.t. \( (f - h)(z) \leq (f - h)(w) \) for all \( z \in \overline{D(P, r)} \).

However, \( f(w) \leq h(w) \), so \( (f - h)(w) \leq 0 \). So any \( z \in \overline{D(P, r)} \), \( (f - h)(z) \leq 0 \) or \( f(z) \leq h(z) \). Thus \( f \) is subharmonic.
(b)
Since $f$ is $C^2$, $f + \varepsilon |z|^2$ is $C^2$. Thus by (a), $f + \varepsilon |z|^2$ is subharmonic. Let $D(P, r) \subset U$, Thus

$$f(P) + \varepsilon |P|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} [f(P + re^{i\theta}) + \varepsilon |P + re^{i\theta}|^2]d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})d\theta + \frac{\varepsilon}{2\pi} \int_0^{2\pi} |P + re^{i\theta}|^2 d\theta$$

Let $\varepsilon \to 0$, Then

$$f(p) \leq \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})d\theta + \lim_{\varepsilon \to 0} \frac{\varepsilon}{2\pi} \int_0^{2\pi} |P + re^{i\theta}|^2 d\theta$$

Then we have

$$f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})d\theta$$

So $f$ is subharmonic.

Problem (7.72) Solution:
By the result from problem 41, we know that $\Delta f \geq 0$. So we have:

$$M'(r) = \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dr} f(re^{i\theta})d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f_r(re^{i\theta})e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{\partial D(0,r)} f_r(z)\frac{1}{r}dz$$

$$= \frac{1}{2\pi i} \int_{\partial D(0,r)} \nabla f(z) \cdot \overrightarrow{n} \frac{dz}{r}$$

$$= \frac{1}{2\pi i} \int_{D(0,r)} \Delta f(z) dz$$

$$\geq \frac{1}{2\pi i} \int_{D(0,r)} 0 dz$$

$$= 0$$

Then we know $M(r)$ is nondecreasing function of $r$.■