Solution of Homework 6

Problem (7.45):
Solution:
Let $P \in \partial U$. By example 7.7.12, it is sufficient to prove that locally each boundary point has a barrier. Since the domain lies only on one side of the curve at $P$, there is a line segment emanating from the boundary point which is exterior to the domain. Hence we prove that $P$ has a barrier and we are done.

Problem (7.51):
Solution:
Suppose $U = C \setminus \{0\}$. We know that $U$ does not have a barrier at 0 and $\partial U = \{0\}$. So we know that $U$ has no barrier at any of its boundary points.

Now suppose $U$ is a bounded domain. Use the way in Example 7.7.10, we can prove that there is a point in $\partial U$ that has a barrier.

Problem (7.52)
Solution:
There exist a harmonic function $u(z)$ such that $u(z) = \log \phi(z)$ at $\partial U$. Since $U$ is simply connected domain, then there exist a $v(z)$ such that $g(z) = u(z) + iv(z)$ is holomorphic. Then $f(z) = \exp^{g(z)} = \exp^{u(z)} \exp^{iv(z)}$ is holomorphic in $U$, continuous on $\overline{U}$. So $|f|$ is continuous on $\overline{U}$ and $|f|_{\partial U} = \phi$. 

Problem (7.53(a))
Solution:
We know that \( g_1(z) = \frac{z^2 - 1}{z^2 + 1} \) is a conformal mapping on the first quadrant to the unit disc.

And
\[
u(z) = u(g_1^{-1}(g_1(z))) = \frac{1}{2\pi} \int_{0}^{2\pi} \phi(g_1^{-1}(e^{i\theta})) \frac{1 - |g_1(z)|^2}{|g_1(z) - e^{i\theta}|^2} d\theta
\]

\]

Problem (7.53(b))
Solution:
\( g_2(z) = \frac{(1+z)^2 - i}{(1+z)^2 + 1} \) is a conformal mapping on upper half of the unit disc with the boundary \( \phi \) to \( D(0, 1) \).

And
\[
u(z) = u(g_2^{-1}(g_2(z))) = \frac{1}{2\pi} \int_{0}^{2\pi} \phi(g_2^{-1}(e^{i\theta})) \frac{1 - |g_2(z)|^2}{|g_2(z) - e^{i\theta}|^2} d\theta
\]

\]

Problem (7.53(c))
Solution: \( g_3(z) = \frac{e^{i\pi z} - z}{e^{i\pi z} + z} \) is a conformal mapping on upper half of the unit disc with the boundary \( \phi \) to \( D(0, 1) \).

And
\[
u(z) = u(g_3^{-1}(g_3(z))) = \frac{1}{2\pi} \int_{0}^{2\pi} \phi(g_3^{-1}(e^{i\theta})) \frac{1 - |g_3(z)|^2}{|g_3(z) - e^{i\theta}|^2} d\theta
\]

\]
Problem (7.54) Solution:
It is easy to see that the result is holds when $\phi$ is linear.

Suppose $\phi$ is a convex function on $\mathbb{R}$. Thus $\phi$ is the pointwise maximum of its supporting tangent lines. So

$$\phi(x) = \max_{x_0 \in [a,b]} (\phi(x_0) + \phi'(x_0)(x - x_0))$$

Hence we have

$$\phi\left(\frac{1}{b-a} \int_a^b f(t) \, dt\right) \geq \max_{x_0 \in [a,b]} (\phi(x_0) + \phi'(x_0) \left(\frac{1}{b-a} \int_a^b f(t) \, dt - x_0\right))$$

$$= \max_{x_0 \in [a,b]} \left(\frac{1}{b-a} \int_a^b \left[\phi(x_0) + \phi'(x_0)(f(t) - x_0)\right] \, dt\right)$$

$$\geq \frac{1}{b-a} \int_a^b \max_{x_0 \in [a,b]} \left[\phi(x_0) + \phi'(x_0)(f(t) - x_0)\right] \, dt$$

$$\geq \frac{1}{b-a} \int_a^b (\phi \circ f)(t) \, dt$$