HYPERBOLICITY OF THE TRACE MAP FOR THE WEAKLY COUPLED FIBONACCI HAMILTONIAN

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Abstract. We consider the trace map associated with the Fibonacci Hamiltonian as a diffeomorphism on the invariant surface associated with a given coupling constant and prove that the non-wandering set of this map is hyperbolic if the coupling is sufficiently small. As a consequence, for these values of the coupling constant, the local and global Hausdorff dimension and the local and global box counting dimension of the spectrum of the Fibonacci Hamiltonian all coincide and are smooth functions of the coupling constant.

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1. Introduction

The Fibonacci Hamiltonian is the most prominent model in the study of electronic properties of quasicrystals. It is given by the discrete one-dimensional Schrödinger operator

$$[H_{V,\omega}u](n) = u(n+1) + u(n-1) + V\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)u(n),$$

where $V > 0$ is the coupling constant, $\alpha = \frac{\sqrt{5}-1}{2}$ is the frequency, and $\omega \in [0,1)$ is the phase.

This operator family displays a number of interesting phenomena, such as Cantor spectrum of zero Lebesgue measure [S89] and purely singular continuous spectral measure for all phases [DL]. Moreover, it was recently shown that is also gives rise to anomalous transport [DT]. We refer the reader to the survey articles [D00, D07, S95] for further information and references.

Already the earliest papers on this model, [KKT, OPRSS], realized the importance of a certain renormalization procedure in its study. This led in particular to a consideration of the following dynamical system, the so-called trace map,

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \ T(x, y, z) = (2xy - z, x, y),$$

whose properties are closely related to all the spectral properties mentioned above. The existence of the trace map and its connection to spectral properties of the operators is a consequence of the invariance of the potential under a substitution rule. This works in great generality; see the surveys mentioned above and references therein. In the Fibonacci case, the existence of a first integral is an additional useful property, which allows one to restrict T to invariant surfaces.

Fix some coupling constant V. For a complete spectral study of the operator family $\{H_{V,\omega}\}_{\omega\in[0,1)}$, it suffices to study T on a single invariant surface S_V. This phenomenon is a peculiarity of the choice of the model (a discrete Schrödinger operator in math terminology or an on-site lattice model in physics terminology) and does not follow solely from the symmetries coming from the invariance under the Fibonacci substitution. For example, in continuum analogs of the Fibonacci Hamiltonian, the invariant surface will in general be energy-dependent.

Let us denote the restriction of T to the invariant surface S_V by T_V. As we will discuss in more detail below, it is of interest to study the non-wandering set of this surface diffeomorphism because it is closely connected to the spectrum of $H_{V,\omega}$.[1] This correspondence in turn allows one to show that the spectrum has zero Lebesgue measure. It is then natural to investigate its fractal dimension. A number of papers have studied this problem; for example, [DEGT, LW, Ra]. As pointed out in [DEGT], the work of Casdagli, [Cas], has very important consequences for the fractal dimension of the spectrum as a function of V. Casdagli studied the map T_V and proved, for $V \geq 16$, that the non-wandering set is hyperbolic. Combining this with results in hyperbolic dynamics, it follows that the local and global Hasudorff and box counting dimensions of the spectrum all coincide and are smooth functions of V. This result was crucial for the work [DEGT], which determined the exact asymptotic behavior of this function of V as V tends to infinity. It was shown that

$$\lim_{V\to\infty} \dim \sigma(H_{V,\omega}) \cdot \log V = \log(1+\sqrt{2}).$$

Of course, the asymptotic behavior of the dimension of the spectrum as V approaches zero is of interest as well. Given the discussion above, the natural first step is to prove the analogue of Casdagli's result at small coupling. This is exactly what we do in this paper. We will show that, for V sufficiently small, the non-wandering set of T_V is hyperbolic and hence we obtain the same consequences for the dimension of the spectrum as those mentioned above in this coupling regime.

The structure of the paper is as follows. Section 2 gives a more explicit description of the previous results on the Fibonacci trace map, recalls some useful general results from hyperbolic dynamics, and states the main result of the paper — the hyperbolicity of the non-wandering set of the trace map for sufficiently small coupling V. Sections 3–6 contain the proof of this result. More precisely, Section 3 contains a discussion of the case $V = 0$, Section 4 studies the dynamics of the trace map near a singular point and formulates the crucial Proposition 1. Section 5 shows how the main result follows from it, and finally, Section 6 contains a proof of Proposition 1.

After this paper was finished we learned that Serge Cantat provided a proof of uniform hyperbolicity of the trace map for all non-zero values of the coupling constant [Can]. Our results were obtained independently and we use completely different methods.

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2. Background and Main Result

In this section we expand on the introduction and state definitions and previous results more carefully. This will eventually lead us to the statement of our main result in Theorem 3 below.

[1]By a strong convergence argument, it follows that the spectrum of $H_{V,\omega}$ does not depend on ω. It does, however, depend on V.

2.1. **Description of the Trace Map and Previous Results.** The main tool that we are using here is the so called *trace map*. It was originally introduced in [K, KKT]; further useful references include [BGJ, BR, HM, Ro]. Let us quickly recall how it arises from the substitution invariance of the Fibonacci potential; see [S87] for detailed proofs of some of the statements below.

The one step transfer matrices associated with the difference equation $H_{V,\omega}u = Eu$ are given by

$$T_{V,\omega}(m, E) = \begin{pmatrix} E - V\chi_{[1-\alpha,1)}(m\alpha + \omega \bmod 1) & -1 \\ 1 & 0 \end{pmatrix}.$$

Denote the Fibonacci numbers by $\{F_k\}$, that is, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k_1}$ for $k \ge 1$. Then, one can show that the matrices

$$M_{-1}(E) = \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix}, \quad M_0(E) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$M_k(E) = T_{V,0}(F_k, E) \times \cdots \times T_{V,0}(1, E) \quad \text{for } k \ge 1$$

obey the recursive relations

$$M_{k+1}(E) = M_{k-1}(E) M_k(E)$$

for $k \ge 0$. Passing to the variables

$$x_k(E) = \frac{1}{2} \mathrm{Tr} M_k(E),$$

this in turn implies

$$x_{k+1}(E) = 2x_k(E)x_{k-1}(E) - x_{k-2}(E).$$

These recursion relations exhibit a conserved quantity; namely, we have

$$x_{k+1}(E)^2 + x_k(E)^2 + x_{k-1}(E)^2 - 2x_{k+1}(E)x_k(E)x_{k-1}(E) - 1 = \frac{V^2}{4}$$

for every $k \ge 0$.

Given these observations, it is then convenient to introduce the *trace map*

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \ T(x, y, z) = (2xy - z, x, y).$$

The following function

$$G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

is invariant under the action of T,[2] and hence T preserves the family of cubic surfaces[3]

$$S_V = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz = 1 + \frac{V^2}{4} \right\}.$$

Plots of the surfaces $S_{0.01}$, $S_{0.1}$, $S_{0.2}$, and $S_{0.5}$ are given in Figures 1–4, respectively.

Denote by ℓ_V the line

$$\ell_V = \left\{ \left(\frac{E - V}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}.$$

It is easy to check that $\ell_V \subset S_V$.

Sütő proved the following central result in [S87].

[2] This invariant is often called the Fricke-Vogt Invariant.

[3] The surface S_0 is called the *Cayley cubic*.

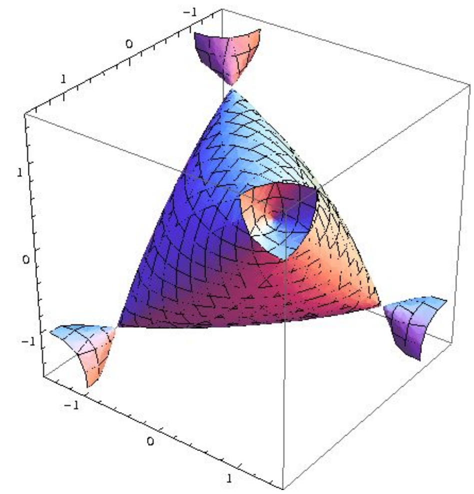

Figure 1. The surface $S_{0.01}$.

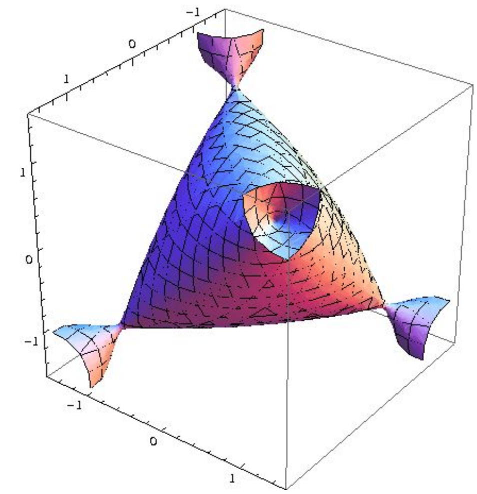

Figure 2. The surface $S_{0.1}$.

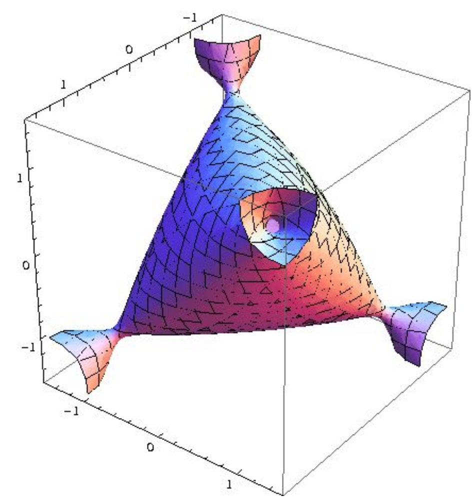

Figure 3. The surface $S_{0.2}$.

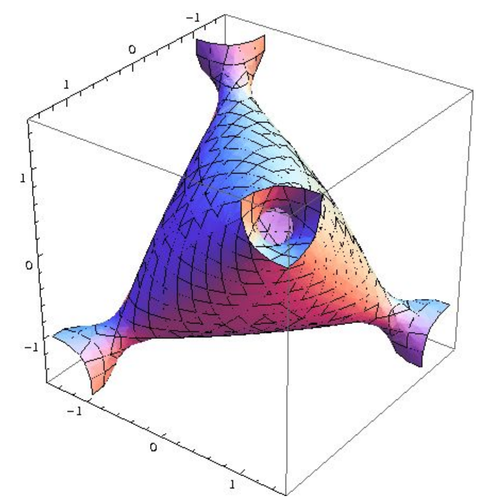

Figure 4. The surface $S_{0.5}$.

Theorem 1 (Sütő 1987)**.** *An energy E belongs to the spectrum of $H_{V,\omega}$ if and only if the positive semiorbit of the point $(\frac{E-V}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.*

It is of course natural to consider the restriction T_V of the trace map T to the invariant surface S_V. That is, $T_V : S_V \to S_V$, $T_V = T|_{S_V}$. Denote by Ω_V the set of points in S_V whose full orbits under T_V are bounded. *A priori* the set of bounded orbits of T_V could be different from the non-wandering set[4] of T_V, but

[4] A point $p \in M$ of a diffeomorphism $f : M \to M$ is wandering if there exists a neighborhood $O(p) \subset M$ such that $f^k(O) \cap O = \emptyset$ for any $k \in \mathbb{Z} \backslash 0$. A non-wandering set of f is a set of points that are not wandering.

our construction of the Markov partition and analysis of the behavior of T_V near singularities show that in our case these two sets do coincide. Notice that this is parallel to the construction of the symbolic coding in [Cas].

Let us recall that an invariant closed set Λ of a diffeomorphism $f : M \to M$ is *hyperbolic* if there exists a splitting of a tangent space $T_xM = E^u_x \oplus E^u_x$ at every point $x \in \Lambda$ such that this splitting is invariant under Df and the differential Df exponentially contracts vectors from stable subspaces $\{E^s_x\}$ and exponentially expands vectors from unstable subspaces $\{E^u_x\}$. A hyperbolic set Λ of a diffeomorphism $f : M \to M$ is *locally maximal* if there exists a neighborhood $U(\Lambda)$ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

The second central result about the trace map we wish to recall is due to Casdagli; see [Cas].

Theorem 2 (Casdagli 1986)**.** *For $V \geq 16$, the set Ω_V is a locally maximal hyperbolic set of $T_V : S_V \to S_V$. It is homeomorphic to a Cantor set.*

2.2. **Some Properties of Locally Maximal Hyperbolic Invariant Sets of Surface Diffeomorphisms.** Given Theorem 2, several general results apply to the trace map of the strongly coupled Fibonacci Hamiltonian. Let us recall some of these results that yield interesting spectral consequences, which are discussed below.

Consider a locally maximal invariant transitive hyperbolic set $\Lambda \subset M$, $\dim M = 2$, of a diffeomorphism $f \in \text{Diff}^r(M), r \geq 1$. We have $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U(\Lambda))$ for some neighborhood $U(\Lambda)$. Assume also that $\dim E^u = \dim E^s = 1$. Then, the following properties hold.

2.2.1. *Stability.* There is a neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ of the map f such that for every $g \in \mathcal{U}$ the set $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g(U(\Lambda))$ is a locally maximal invariant hyperbolic set set of g. Moreover, there is a homeomorphism $h : \Lambda \to \Lambda_g$ that conjugates $f|_\Lambda$ and $g|_{\Lambda_g}$, that is, the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{f|_\Lambda} & \Lambda \\ h\downarrow & & \downarrow h \\ \Lambda_g & \xrightarrow{g|_{\Lambda_g}} & \Lambda_g \end{array}$$

2.2.2. *Invariant Manifolds.* For $x \in \Lambda$ and small $\varepsilon > 0$, consider the local stable and unstable sets

$$W^s_\varepsilon(x) = \{w \in M : d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \geq 0\},$$
$$W^u_\varepsilon(x) = \{w \in M : d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \leq 0\}.$$

If $\varepsilon > 0$ is small enough, these sets are embedded C^r-disks with $T_xW^s_\varepsilon(x) = E^s_x$ and $T_xW^u_\varepsilon(x) = E^u_x$. Define the (global) stable and unstable sets as

$$W^s(x) = \bigcup_{n\in\mathbb{N}} f^{-n}(W^s_\varepsilon(x)), \quad W^u(x) = \bigcup_{n\in\mathbb{N}} f^n(W^u_\varepsilon(x)).$$

Define also

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x) \quad \text{and} \quad W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x).$$

2.2.3. *Invariant Foliations.* A stable foliation for Λ is a foliation $\mathcal{F}^s$ of a neighborhood of Λ such that

(a) for each $x \in \Lambda$, $\mathcal{F}(x)$, the leaf containing x, is tangent to E^s_x,
(b) for each x sufficiently close to Λ, $f(\mathcal{F}^s(x)) \subset \mathcal{F}^s(f(x))$.

An unstable foliation $\mathcal{F}^u$ can be defined in a similar way.

For a locally maximal hyperbolic set $\Lambda \subset M$ of a C^1-diffeomorphism $f : M \to M$, $\dim M = 2$, stable and unstable C^0 foliations with C^1-leaves can be constructed [M]. In the case of C^2-diffeomorphism, C^1 invariant foliations exist (see, for example, [PT], Theorem 8 in Appendix 1).

2.2.4. *Local Hausdorff Dimension and Box Counting Dimension.* Consider, for $x \in \Lambda$ and small $\varepsilon > 0$, the set $W^u_\varepsilon(x) \cap \Lambda$. Its Hausdorff dimension does not depend on $x \in \Lambda$ and $\varepsilon > 0$, and coincides with its box counting dimension (see [MM, T]):

$$\dim_H W^u_\varepsilon(x) \cap \Lambda = \dim_B W^u_\varepsilon(x) \cap \Lambda.$$

In a similar way,

$$\dim_H W^s_\varepsilon(x) \cap \Lambda = \dim_B W^s_\varepsilon(x) \cap \Lambda.$$

Denote $h^s = \dim_H W^s_\varepsilon(x) \cap \Lambda$ and $h^u = \dim_H W^u_\varepsilon(x) \cap \Lambda$. We will say that h^s and h^u are the *local stable and unstable Hausdorff dimensions* of Λ.

For properly chosen small $\varepsilon > 0$ the sets $W^u_\varepsilon(x) \cap \Lambda$ and $W^s_\varepsilon(x) \cap \Lambda$ are dynamically defined Cantor sets (see [PT1] for definitions and proof), and this implies, in particular, that

$$h^s < 1 \quad \text{and} \quad h^u < 1,$$

see, for example, Theorem 14.5 in [P].

2.2.5. *Global Hausdorff Dimension.* The Hausdorff dimension of Λ is equal to its box counting dimension, and

$$\dim_H \Lambda = \dim_B \Lambda = h^s + h^u;$$

see [MM, PV].

2.2.6. *Continuity of the Hausdorff Dimension.* The local Hausdorff dimensions $h^s(\Lambda)$ and $h^u(\Lambda)$ depend continuously on $f : M \to M$ in the C^1-topology; see [MM, PV]. Therefore, $\dim_H \Lambda_f = \dim_B \Lambda_f = h^s(\Lambda_f) + h^u(\Lambda_f)$ also depends continuously on f in the C^1-topology. Moreover, for a C^r diffeomorphism $f : M \to M$, $r \geq 2$, the Hausdorff dimension of a hyperbolic set Λ_f is a C^{r-1} function of f; see [Ma].

Remark 2.1. *For hyperbolic sets in dimension greater than two, many of these properties do not hold in general; see* [P] *for more details.*

2.3. **Implications for the Trace Map and the Spectrum.** Due to Theorem 2, for every $V \geq 16$, all the properties from the previous subsection can be applied to the hyperbolic set Ω_λ of the trace map $T_V : S_V \to S_V$.

Moreover, the results in [Cas, Section 2] imply the following statement.

Lemma 2.2. *For $V \geq 16$ and every $x \in \Omega_V$, the stable manifold $W^s(x)$ intersects the line ℓ_V transversally.*

The existence of a C^1-foliation $\mathcal{F}^s$ allows one to locally consider the set $W^s(\Omega_V) \cap \ell_V$ as a C^1-image of the set $W^u_\varepsilon(x) \cap \Omega_V$. Due to Theorem 1, this implies the following properties of the spectrum $\sigma(H_{V,\omega})$ for $V \geq 16$:

Corollary 1. *For $V \geq 16$, the following statements hold:*

(i) *The spectrum $\sigma(H_{V,\omega})$ depends continuously on V in the Hausdorff metric.*
(ii) *For every small $\varepsilon > 0$ and every $x \in \sigma(H_{V,\omega})$, we have*

$$\begin{aligned}\dim_H\left((x-\varepsilon, x+\varepsilon) \cap \sigma(H_{V,\omega})\right) &= \dim_B\left((x-\varepsilon, x+\varepsilon) \cap \sigma(H_{V,\omega})\right) \\ &= \dim_H \sigma(H_{V,\omega}) \\ &= \dim_B \sigma(H_{V,\omega}).\end{aligned}$$

(iii) *The Hausdorff dimension $\dim_H \sigma(H_{V,\omega})$ is a C^∞-function of V.*

2.4. **Hyperbolicity of the Trace Map for Small Coupling.** We are now in a position to state the main result of this paper.

Theorem 3. *There exists $V_0 > 0$ such that for every $V \in (0, V_0)$, the following properties hold.*

(i) *The non-wandering set $\Omega_V \subset S_V$ of the map $T_V : S_V \to S_V$ is hyperbolic.*
(ii) *The non-wandering set $\Omega_V \subset S_V$ is homeomorphic to a Cantor set, and $T_V|_{\Omega_V}$ is conjugated to a topological Markov chain $\sigma_A : \Sigma_A^6 \to \Sigma_A^6$ with the matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(iii) *For every $x \in \Omega_V$, the stable manifold $W^s(x)$ intersects the line ℓ_V transversally.*

As before, we obtain the following consequences:

Corollary 2. *With $V_0 > 0$ from Theorem 3, the following statements hold for $V \in (0, V_0)$:*

(i) *The spectrum $\sigma(H_{V,\omega})$ depends continuously on V in the Hausdorff metric.*
(ii) *For every small $\varepsilon > 0$ and every $x \in \sigma(H_{V,\omega})$, we have*

$$\begin{aligned}\dim_H\left((x-\varepsilon, x+\varepsilon) \cap \sigma(H_{V,\omega})\right) &= \dim_B\left((x-\varepsilon, x+\varepsilon) \cap \sigma(H_{V,\omega})\right) \\ &= \dim_H \sigma(H_{V,\omega}) \\ &= \dim_B \sigma(H_{V,\omega}).\end{aligned}$$

(iii) *The Hausdorff dimension $\dim_H \sigma(H_{V,\omega})$ is a C^∞-function of V, and is strictly smaller than one.*

We expect these properties to be of similar importance in a study of the asymptotic behavior of the fractal dimension of the spectrum as $V \to 0$ as was the case in the large coupling regime.

3. Properties of the Trace Map for $V = 0$

Up to this point, we only considered the case $V > 0$. Since we will regard the case of small positive V as a small perturbation of the case $V = 0$, we will also include the latter case in our considerations. In fact, this section is devoted to the study of this "unperturbed case."

Denote by $\mathbb{S}$ the part of the surface S_0 inside of the cube $\{|x| \le 1, |y| \le 1, |z| \le 1\}$. The surface $\mathbb{S}$ is homeomorphic to S^2, invariant, smooth everywhere except at the four points $P_1 = (1,1,1)$, $P_1 = (1,-1,-1)$, $P_1 = (-1,1,-1)$, and $P_1 = (-1,-1,1)$, where $\mathbb{S}$ has conic singularities, and the trace map T restricted to $\mathbb{S}$ is a factor of a hyperbolic automorphism of a two torus:

$$\mathcal{A}(\theta,\varphi) = (\theta+\varphi,\theta) \ (\mathrm{mod}\ 1).$$

The semiconjugacy is given by the map

$$F : (\theta,\varphi) \mapsto (\cos 2\pi(\theta+\varphi), \cos 2\pi\theta, \cos 2\pi\varphi).$$

The map $\mathcal{A}$ is hyperbolic, and is given by a matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalues

$$\mu = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad -\mu^{-1} = \frac{1-\sqrt{5}}{2}.$$

Let us denote by $\mathbf{v}^u, \mathbf{v}^u \in \mathbb{R}^2$ the unstable and stable eigenvectors of A:

$$A\mathbf{v}^u = \mu\mathbf{v}^u, \ A\mathbf{v}^s = -\mu^{-1}\mathbf{v}^s, \ \|\mathbf{v}^u\| = \|\mathbf{v}^s\| = 1.$$

Fix some small $\zeta > 0$ and define the stable (resp., unstable) cone fields on $\mathbb{R}^2$ in the following way:

$$K_p^s = \{\mathbf{v} \in T_p\mathbb{R}^2 | \mathbf{v} = v^u\mathbf{v}^u + v^s\mathbf{v}^s, \ |v^s| > \zeta^{-1}|v^u|\},$$
$$K_p^u = \{\mathbf{v} \in T_p\mathbb{R}^2 | \mathbf{v} = v^u\mathbf{v}^u + v^s\mathbf{v}^s, \ |v^u| > \zeta^{-1}|v^s|\}. \tag{1}$$

These cone fields are invariant:

$$\forall\ \mathbf{v} \in K_p^u \quad A\mathbf{v} \in K_{A(p)}^u,$$
$$\forall\ \mathbf{v} \in K_p^s \quad A^{-1}\mathbf{v} \in K_{A^{-1}(p)}^s.$$

Also, the iterates of the map A expand vectors from the unstable cones, and the iterates of the map A^{-1} expand vectors from the stable cones:

$$\forall\ \mathbf{v} \in K_p^u \quad \forall\ n \in \mathbb{N} \quad |A^n\mathbf{v}| > \frac{1}{\sqrt{1+\zeta^2}}\lambda^n|\mathbf{v}|,$$
$$\forall\ \mathbf{v} \in K_p^s \quad \forall\ n \in \mathbb{N} \quad |A^{-n}\mathbf{v}| > \frac{1}{\sqrt{1+\zeta^2}}\lambda^n|\mathbf{v}|.$$

The families of cones $\{K^s\}$ and $\{K^u\}$ invariant under $\mathcal{A}$ can be also considered on $\mathbb{T}^2$.

The differential of the semiconjugacy F sends these cone families to stable and unstable cone families on $\mathbb{S}\backslash\{P_1,P_2,P_3,P_4\}$. Let us denote these images by $\{\mathcal{K}^s\}$ and $\{\mathcal{K}^u\}$.

Lemma 3.1. *The differential of the semiconjugacy DF induces a map of the unit bundle of $\mathbb{T}^2$ to the unit bundle of $\mathbb{S}\backslash\{P_1,P_2,P_3,P_4\}$. The derivatives of the restrictions of this map to a fiber are uniformly bounded. In particular, the sizes of cones in families $\{\mathcal{K}^s\}$ and $\{\mathcal{K}^u\}$ are uniformly bounded away from zero.*

Proof. Choose small enough neighborhoods $U_1(P_1), U_2(P_2), U_3(P_3)$, and $U_4(P_4)$ in $\mathbb{S}$. The complement

$$\hat{\mathbb{S}} = \mathbb{S} \setminus \Big(\bigcup_{j=1}^{4} U_j(P_j) \Big)$$

is compact, so $F^{-1}(\hat{\mathbb{S}})$ is also compact, and the action of DF on a fiber of the unit bundle over points of $F^{-1}(\hat{\mathbb{S}})$ has uniformly bounded derivatives.

Due to the symmetries of the trace map (see, for example, [K] for the detailed description of the symmetries of the trace map) and the semiconjugacy F, it is enough to consider a neighborhood $U_1(P_1)$. Hence, it is enough to consider the map F in a neighborhood of the point $(0,0)$.

The differential of F has the form

$$DF = -2\pi \begin{pmatrix} \sin 2\pi(\theta+\varphi) & \sin 2\pi(\theta+\varphi) \\ \sin 2\pi\theta & 0 \\ 0 & \sin 2\pi\varphi \end{pmatrix}.$$

If (θ,φ) is in a small neighborhood of $(0,0)$, then

$$DF \sim -4\pi^2 \begin{pmatrix} \theta+\varphi & \theta+\varphi \\ \theta & 0 \\ 0 & \varphi \end{pmatrix}.$$

Therefore, up to a multiplicative constant and higher order terms, the image of the vector $(1,0)$ under DF is $(\theta+\varphi,\theta,0)$, and the image of the vector $(0,1)$ is $(\theta+\varphi,0,\varphi)$. In order to estimate the derivative of the projective action of DF on a fiber over a point (θ,φ) it is enough to estimate the angle between images of basis vectors, and the ratio of the lengthes of the images of these vectors.

If α is an angle between $DF_{(\theta,\varphi)}(1,0)$ and $DF_{(\theta,\varphi)}(0,1)$, then

$$\cos\alpha \sim \frac{(\theta+\varphi)^2}{\sqrt{(\theta+\varphi)^2+\theta^2}\cdot\sqrt{(\theta+\varphi)^2+\varphi^2}} = \frac{1}{\sqrt{1+\frac{\theta^2+\varphi^2}{(\theta+\varphi)^2}+\frac{\theta^2\varphi^2}{(\theta+\varphi)^4}}}$$

We have

$$\frac{\theta^2+\varphi^2}{(\theta+\varphi)^2} \ge \frac{1}{2}, \quad \text{and} \quad \frac{\theta^2\varphi^2}{(\theta+\varphi)^4} \ge 0,$$

so $\cos\alpha \le \sqrt{\frac{2}{3}} + 0.001 < 1$ if (θ,φ) is close enough to $(0,0)$.

Now let us estimate the ratio of the lengths of $DF_{(\theta,\varphi)}(1,0)$ and $DF_{(\theta,\varphi)}(0,1)$. Up to higher order terms it is equal to

$$\frac{\sqrt{\varphi^2+2\theta^2+2\theta\varphi}}{\sqrt{2\varphi^2+\theta^2+2\theta\varphi}} = \sqrt{\frac{1+2t^2+2t}{2+t^2+2t}} = \sqrt{\frac{2(t+\frac{1}{2})^2+\frac{1}{2}}{(t+1)^2+1}},$$

where $t = \frac{\theta}{\varphi} \in \mathbb{R}\cup\{\infty\}$, and this function is bounded from above and bounded away from zero. Lemma 3.1 is proved. □

4. The Structure of the Trace Map in a Neighborhood of a Singular Point

Due to the symmetries of the trace map it is enough to consider the dynamics of T in a neighborhood of $P_1 = (1,1,1)$. Let $U \subset \mathbb{R}^3$ be a small neighborhood of P_1 in $\mathbb{R}^3$. Let us consider the set $Per_2(T)$ of periodic points of T of period 2.

Lemma 4.1. *We have*

$$Per_2(T) = \left\{(x,y,z) : x \in (-\infty,\tfrac{1}{2}) \cup (\tfrac{1}{2},\infty),\ y = \frac{x}{2x-1},\ z = x\right\}.$$

Proof. Direct calculation. □

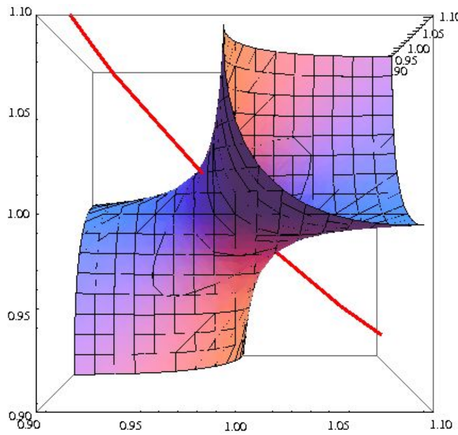

FIGURE 5. $S_{0.1}$ and $Per_2(T)$ near $(1,1,1)$.

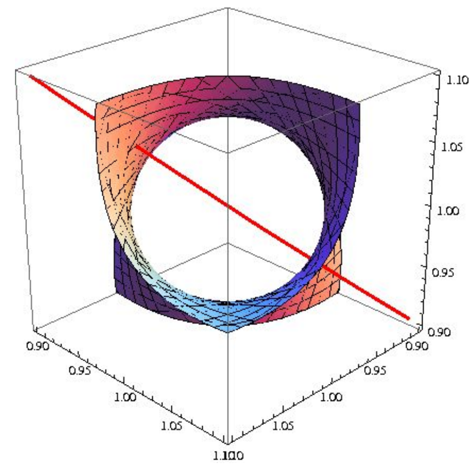

FIGURE 6. $S_{0.2}$ and $Per_2(T)$ near $(1,1,1)$.

Notice that in a neighborhood U the intersection $I \equiv Per_2(T) \cap U$ is a smooth curve that is a normally hyperbolic with respect to T (see, for example, [PT], Appendix 1, for the formal definition of normal hyperbolicity). Therefore the local center-stable manifold $W^{cs}_{loc}(I)$ and the local center-unstable manifold $W^{cu}_{loc}(I)$ defined by

$$W^{cs}_{loc}(I) = \{p \in U : T^n(p) \in U \text{ for all } n \in \mathbb{N}\},$$
$$W^{cu}_{loc}(I) = \left\{p \in U : T^{-n}(p) \in U \text{ for all } n \in \mathbb{N}\right\}$$

are smooth two-dimensional surfaces. Also, the local strong stable manifold $W^{ss}_{loc}(P_1)$ and the local strong unstable manifold $W^{uu}_{loc}(P_1)$ of the fixed point P_1, defined by

$$W^{ss}_{loc}(P_1) = \{p \in W^{cs}_{loc}(I) : T^n(p) \to P_1 \text{ as } n \to +\infty\},$$
$$W^{uu}_{loc}(P_1) = \left\{p \in W^{cu}_{loc}(I) : T^{-n}(p) \to P_1 \text{ as } n \to +\infty\right\},$$

are smooth curves.

Let $\Phi : U \to \mathbb{R}^3$ be a smooth change of coordinates such that $\Phi(P_1) = (0,0,0)$ and

- $\Phi(I)$ is a part of the line $\{x = 0, z = 0\}$;
- $\Phi(W^{cs}_{loc}(I))$ is a part of the plane $\{z = 0\}$;
- $\Phi(W^{cu}_{loc}(I))$ is a part of the plane $\{x = 0\}$;
- $\Phi(W^{ss}_{loc}(P_1))$ is a part of the line $\{y = 0, z = 0\}$;
- $\Phi(W^{uu}_{loc}(P_1))$ is a part of the line $\{x = 0, y = 0\}$.

Denote $f = \Phi \circ T \circ \Phi^{-1}$.

In this case

$$Df(0,0,0) = D(\Phi \circ T \circ \Phi^{-1})(0,0,0) = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is a largest eigenvalue of the differential $DT(P_1) : T_{P_1}\mathbb{R}^3 \to T_{P_1}\mathbb{R}^3$,

$$DT(P_1) = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda = \frac{3+\sqrt{5}}{2} = \mu^2.$$

Let us denote $\mathfrak{S}_V = \Phi(S_V)$. Then away from $(0,0,0)$ the family $\{\mathfrak{S}_V\}$ is a smooth family of surfaces, $\mathfrak{S}_0$ is diffeomorphic to a cone, contains lines $\{y = 0, z = 0\}$ and $\{x = 0, y = 0\}$, and at each non-zero point on those lines it has a quadratic tangency with a horizontal or vertical plane; compare Figures 5 and 6.

We will use variables (x, y, z) for coordinates in $\mathbb{R}^3$. For a point $p \in \mathbb{R}^3$ we will denote its coordinates by (x_p, y_p, z_p).

In order to study the properties of the map f (i.e. of the map T in a small neighborhood of singularities), we need the following statement.

Proposition 1. *Given $C_1 > 0, C_2 > 0, \lambda > 1$, $\varepsilon \in (0, \frac{1}{4})$, $\eta > 0$ there exists $\delta_0 = \delta_0(C_1, C_2, \lambda, \varepsilon)$, $N_0 \in \mathbb{N}, N_0 = N_0(C_1, C_2, \lambda, \varepsilon, \delta_0)$, and $C = C(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ the following holds.*

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2-diffeomorphism such that

(i) $\|f\|_{C^2} \le C_1$;
(ii) *The plane $\{z = 0\}$ is invariant under iterates of f;*
(iii) $\|Df(p) - A\| < \delta$ *for every $p \in \mathbb{R}^3$, where*

$$A = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

is a constant matrix.

Introduce the following cone field in $\mathbb{R}^3$:

$$K_p = \{v \in T_p\mathbb{R}^3,\ \mathbf{v} = \mathbf{v}_{xy} + \mathbf{v}_z : |\mathbf{v}_z| \ge C_2\sqrt{|z_p|}|\mathbf{v}_{xy}|\}. \tag{2}$$

Given a point $p = (x_p, y_p, z_p)$ such that $0 < z_p < 1$ denote by $N = N(p)$ the smallest integer $N \in \mathbb{N}$ such that $f^N(p)$ has z-coordinate larger that 1.

If $N(p) \ge N_0$ (i.e. if z_p is small enough) then

$$|Df^N(\mathbf{v})| \ge \lambda^{\frac{N}{2}(1-4\varepsilon)}|\mathbf{v}| \quad \text{for any} \quad \mathbf{v} \in K_p, \tag{3}$$

and if $Df^N(\mathbf{v}) = \mathbf{u} = \mathbf{u}_{xy} + \mathbf{u}_z$, then

$$|\mathbf{u}_{xy}| < 2\delta^{1/2}|\mathbf{u}_z|. \tag{4}$$

Moreover, if $|\mathbf{v}_z| \ge \eta|\mathbf{v}_{xy}|$ then

$$|Df^k(\mathbf{v})| \ge C\lambda^{\frac{k}{2}(1-4\varepsilon)}|\mathbf{v}| \quad \text{for each} \quad k = 1, 2, \ldots, N. \tag{5}$$

Returning to our specific situation at hand, if, for a given δ, the neighborhood U is small enough, then at every point $P \in U$ the differential $D(\Phi \circ T \circ \Phi^{-1})(P)$ satisfies the condition (iii) of Proposition 1. Also, since the tangency of $\mathfrak{S}_0$ with the horizontal plane is quadratic, there exists $C_2 > 0$ such that every vector tangent to $\mathfrak{S}_0$ from the cone $D\Phi(\mathcal{K}^u)$ also belongs to the cone (2). The same holds for vectors tangent to $\mathfrak{S}_V$ from continuations of cones $D\Phi(\mathcal{K}^u)$ if V is small enough. Therefore, Proposition 1 can be applied to all those vectors.

We postpone the proof of Proposition 1 to Section 6 and first show how to use it to prove Theorem 3.

5. Proof of Theorem 3 Assuming Proposition 1

In order to prove hyperbolicity of Ω_V we construct only the unstable cone field and prove the unstable cone condition. Due to the symmetry of the trace map, the stable cones can be constructed in the same way.

Let U' be a neighborhood of $\{P_1, P_2, P_3, P_4\}$ where the results of the previous section can be applied. Since $\mathbb{S}\backslash U'$ is compact, $F^{-1}(\mathbb{S}\backslash U')$ is also compact. Denote

$$\widetilde{C} = 2 \max_{p\in\mathbb{S}\backslash U'} \left\{\|DF^{-1}(p)\|, \|DF(F^{-1}(p))\|\right\} < \infty.$$

Take any $p \in \mathbb{S}\backslash U'$ and any $\mathbf{v} \in T_p\mathbb{S}, \mathbf{v} \in \mathcal{K}^u$. If $T^n(p) \in \mathbb{S}\backslash U'$ then

$$\|DT^n(\mathbf{v})\| = \|D(F \circ \mathcal{A}^n \circ F^{-1})(\mathbf{v})\| \geq \widetilde{C}^{-2}\mu^n\|\mathbf{v}\|. \tag{6}$$

Fix small $\varepsilon > 0$ and $n^* \in \mathbb{N}$ such that $\widetilde{C}^{-2}\mu^{n^*} \geq \mu^{n^*(1-\varepsilon)}$.

Lemma 5.1. *There exists a neighborhood $U^* \subset U'$ of the singular set $\{P_1, P_2, P_3, P_4\}$ such that if $p \notin U^*$ but $T^{-1}(p) \in U^*$, and n_0 is the smallest positive integer such that $T^{n_0}(p) \in U^*$, then the finite orbit $\{p, T(p), \dots, T^{n_0-1}(p)\}$ contains at least n^* points outside of U'.*

Proof. Take a point $p_1 \in W^{uu}_{loc}(P_1) \subset U$, denote $p_2 = T(p_1)$ and consider the closed arc $J \subset W^{uu}(P_1)$ between points p_1 and p_2. For any point $p \in J$, denote by $m(p)$ the smallest number $m \in \mathbb{N}$ such that the finite orbit $\{p, T(p), \dots, T^m(p)\}$ contains n^* points outside of $\overline{U'}$. Notice that function $m(p)$ is upper semicontinuous, and therefore (due to compactness of J) is bounded. Let $M \in \mathbb{N}$ be an upper bound. The set $\bigcup_{i=0}^M T^i(J)$ is compact and does not contain the singularities P_1, P_2, P_3 and P_4. Therefore there exists $\xi > 0$ so small that if $\operatorname{dist}(q, J) < \xi$, then the distance between any of the first M iterates of q and any of the points P_1, P_2, P_3, P_4 is greater than ξ, and the finite orbit $\{q, T(q), \dots, T^M(q)\}$ contains at lest n^* points outside of $\overline{U'}$. Now take $\xi' > 0$ so small that any point in ξ'-neighborhood of P_1 whose orbit follows $W^{uu}(P_1)$ towards p_1 hits the ξ-neighborhood of J before leaving U'. Now we can take the ξ'-neighborhood of the set P_1, P_2, P_3, P_4 as U^*. □

For small V, denote by $\mathbb{S}_{V,U^*}$ the bounded component of $S_V\backslash U^*$. The family $\{\mathbb{S}_{V,U^*}\}_{V\in[0,V_0)}$ of surfaces with boundary depends smoothly on the parameter and has uniformly bounded curvature. For small V a projection $\pi_V : \mathbb{S}_{V,U^*} \to \mathbb{S}$ is defined. The map π_V is smooth, and if $p \in \mathbb{S}, q \in \mathbb{S}_{V,U^*}, \pi_V(q) = p$, then $T_p\mathbb{S}$ and T_qS_V are close. Denote by $\mathcal{K}^u_V$ (resp., $\mathcal{K}^s_V$) the image of the cone $\mathcal{K}^u$ (resp., $\mathcal{K}^s$) under the differential of π_V^{-1}.

Our choice of n^* guarantees that the following statement holds.

Lemma 5.2. *There exists $V_0 > 0$ such that for any $V \in [0, V_0)$ the following holds: if $\{q, T(q), T^2(q), \dots, T^n(q)\} \subset \mathbb{S}_{V,U^*}$, $\mathbf{v} \in \mathcal{K}^u_V \subset T_qS_V$, $q, T^n(q) \in \mathbb{S}_{V,U'}$, and $n \geq n^*$, then*

$$\|DT^n(\mathbf{v})\| \geq \mu^{n(1-2\varepsilon)}\|\mathbf{v}\|.$$

Lemma 5.3. *There exist $V_0 > 0$ and $C > 0$ such that for any $V \in [0, V_0)$, we have that if $q \in \mathbb{S}_{V,U'}$, $\mathbf{v} \in \mathcal{K}^u_V \subset T_qS_V$, and $T^n(q) \in \mathbb{S}_{V,U'}$, then*

$$\|DT^n(\mathbf{v})\| \geq C\mu^{n(1-4\varepsilon)}\|\mathbf{v}\|.$$

Proof. Let us split the orbit $\{q, T(q), T^2(q), \dots, T^n(q)\}$ into several intervals

$$\{q, T(q), T^2(q), \dots, T^{k_1-1}(q)\}, \{T^{k_1}(q), \dots, T^{k_2-1}(q)\}, \dots, \{T^{k_s}(q), \dots, T^n(q)\}$$

in such a way that the following properties hold:

(1) for each $i = 1, 2, \dots, s$ the points $T^{k_i-1}(q)$ and $T^{k_i}(q)$ are outside of U';
(2) if $\{T^{k_i}(q), \dots, T^{k_{i+1}-1}(q)\} \cap U^* \neq \emptyset$ then $\{T^{k_i+1}(q), \dots, T^{k_{i+1}-2}(q)\} \subset U'$;
(3) for each $i = 1, 2, \dots, s-1$ we have either $k_{i+1} - k_i \geq n^*$ or $\{T^{k_i}(q), \dots, T^{k_{i+1}-1}(q)\} \cap U^* \neq \emptyset$.

Such a splitting exists due to the choice of U^* above.

The following lemma is a consequence of Lemma 3.1 and property (4) from Proposition 1.

Lemma 5.4. *Suppose $q \in \mathbb{S}_{V,U'}$, $\mathbf{v} \in \mathcal{K}_V^u \subset T_q S_V$, $T(q) \in U'$, and $l \in \mathbb{N}$ is the smallest number such that $T^l(q) \notin U'$. If l is large enough and V is small enough, then*

$$DT^l(q)(\mathbf{v}) \in \mathcal{K}_V^u \subset T_{T^l(q)} S_V.$$

Apply Lemma 5.4 to those intervals in the splitting that are contained in U'. Notice that largeness of l can be provided by the choice of U^*. Together with Proposition 1 applied to these intervals, and Lemma 5.2 applied to intervals that do not intersect U^*, this guarantees uniform expansion of $\mathbf{v}$. The first and the last interval may have length greater than n^*, and then Lemma 5.2 can be applied, or smaller that n^*, but then taking small enough constant C (say, $C < \widetilde{C}^{-2}$) will compensate the lack of uniform expansion on these intervals. Lemma 5.3 is proved. □

For any small $V > 0$ there exists $\eta_V > 0$ ($\eta_V \to 0$ as $V \to 0$) such that if $p \notin U'$, $T(p) \in U'$, and $\mathbf{v} \in \mathcal{K}_V^u(p)$ then for the vector $\mathbf{w} \equiv D\Phi^{-1}(p)(\mathbf{v}), \mathbf{w} = \mathbf{w}_z + \mathbf{w}_{xy}$, we have $|\mathbf{w}_z| \geq \eta_V |\mathbf{w}_{xy}|$. Application of Proposition 1 together with Lemma 5.3 proves uniform expansion of vectors from $\mathcal{K}_V^u$. Due to the symmetries of the trace map, all vectors from $\mathcal{K}_V^s$ are expanded by iterates of T^{-1}. Thus, hyperbolicity of the set Ω_V follows and Theorem 3 (i) is proved.

The Markov partition for $T|_{\mathbb{S}}$ (i.e., for $V = 0$) was presented explicitly by Casdagli in [Cas, Section 2]; compare Figure 5.

Since the Markov partition is formed by finite pieces of strong stable and strong unstable manifolds of the periodic points of T and these manifolds depend smoothly on the parameter V, there exists a Markov partition for Ω_V with the same matrix as for $V = 0$ (see [PT, Appendix 2] for more details on Markov partitions for 2-dimensional hyperbolic maps). This establishes Theorem 3 (ii).

Lemma 5.5. *If V is small enough, then the line ℓ_V is transversal to the cone field $\mathcal{K}_V^s$.*

Proof. For $V = 0$ and small enough ζ in (1), this is true since $F^{-1}(l_0) = \{\theta = -\varphi\}$, and the vector $(1, -1)$ is not an eigenvector of A. Therefore this is also true for every sufficiently small V by continuity. □

In order to show that ℓ_V is also transversal to the stable manifolds of Ω_V inside of U^*, let us consider the rectifying coordinates $\Phi : U^* \to \mathbb{R}^3$ again and define the central-unstable cone field in $\Phi(U^*)$:

$$K_p^{cu} = \left\{ \mathbf{v} \in T_p\mathbb{R}^3, \mathbf{v} = \mathbf{v}_x + \mathbf{v}_{yz} : |\mathbf{v}_{yz}| > \zeta^{-1} |\mathbf{v}_x| \right\}.$$

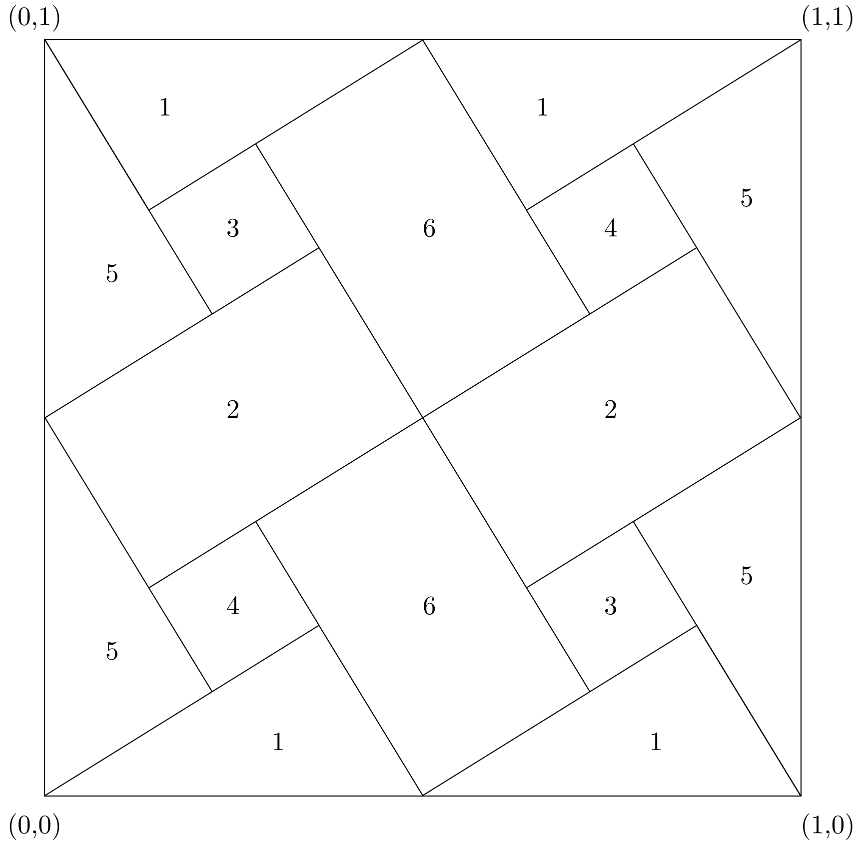

Figure 7. The Markov partition for $T|_{\mathbb{S}}$.

Since $\Phi(l_0)$ is transversal to the plane $\{x = 0\}$, the curve $\Phi(\ell_V)$ is transversal to this invariant cone field if ζ, V, and U^* are small enough. Every stable manifold of Ω_V in the rectifying coordinates is tangent to this central-unstable cone field, and together with Lemma 5.5 this implies that ℓ_V is transversal to stable manifolds of Ω_V. This shows Theorem 3 (iii) and hence concludes the proof of Theorem 3.

6. Proof of Proposition 1

6.1. **Properties of the recurrent sequences.** In this subsection we formulate and prove several lemmas on recurrent sequences that will be used in the next subsection to prove Proposition 1.

Lemma 6.1. *Given $C_1 > 0, C_2 > 0, \lambda > 1$, $\varepsilon \in (0, \frac{1}{4})$, there exists $\delta_0 = \delta_0(C_1, C_2, \lambda, \varepsilon)$ and $N_0 \in \mathbb{N}, N_0 = N_0(C_1, C_2, \lambda, \varepsilon, \delta_0)$ such that for any $\delta \in (0, \delta_0)$ and any $N \geq N_0$ the following holds. Suppose that the sequences $\{d_i\}_{i=0}^N, \{D_i\}_{i=0}^N$*

are defined by the initial conditions

$$\begin{cases} d_0 = 1 \\ D_0 \geq C_2 \frac{1}{(\lambda+\delta)^{N/2}}, \end{cases} \tag{7}$$

and recurrence relations

$$\begin{cases} d_{k+1} = (1+2\delta)d_k + \delta D_k, \\ D_{k+1} = (\lambda - \delta)D_k - C_1 b_k d_k, \end{cases} \tag{8}$$

where

$$b_k = (\lambda - \delta)^{-N+k}.$$

Then

$$\begin{cases} d_N \leq 2\delta^{1/2} D_N \\ D_N \geq D_0 \lambda^{N(1-\varepsilon)} > \lambda^{\frac{N}{2}(1-4\varepsilon)}. \end{cases}$$

Remark 6.2. *Notice that Lemma 6.1 implies also that*

$$\frac{d_N}{D_N} \leq 2\delta^{1/2}.$$

This inequality will allow us to obtain a small cone (of size of order $\delta^{1/2}$) where the image of a vector after leaving a neighborhood of a singularity is located.

Proof of Lemma 6.1. Let us denote

$$\Lambda_- = \lambda^{1-\varepsilon}, \quad \Lambda_+ = \lambda^{1+\varepsilon}.$$

If δ is small enough then $1 < \Lambda_- < \lambda - \delta < \lambda + \delta < \Lambda_+$. We will prove by induction that

$$\begin{cases} d_k \leq (1 + 2\delta + \delta^{1/2}) \max\{d_{k-1}, \delta^{1/2} D_{k-1}\} \\ D_k \geq \Lambda_- D_{k-1} \geq \Lambda_-^k D_0. \end{cases} \tag{9}$$

It is clear that these inequalities for $k = N$ imply Lemma 6.1.

Let us first check the base of induction. If N is large then $D_0 < 1$, so $\max\{d_0, \delta^{1/2} D_0\} = d_0 = 1$. So we have

$$d_1 = (1+2\delta)d_0 + \delta D_0 < 1 + 3\delta < (1 + 2\delta + \delta^{1/2})d_0.$$

This is the first inequality in (9) for $k = 1$.

Also we have

$$\begin{aligned} D_1 &= (\lambda - \delta)D_0 - C_1 b_0 d_0 \\ &= (\lambda - \delta)D_0 \left(1 - \frac{C_1 b_0 d_0}{D_0}\right) \\ &\geq (\lambda - \delta)D_0 \left(1 - \frac{C_1(\lambda+\delta)^{N/2}}{C_2(\lambda-\delta)^N}\right) \\ &> (\lambda - \delta)D_0 \left(1 - \frac{C_1 \Lambda_+^{N/2}}{C_2 \Lambda_-^N}\right) \\ &= (\lambda - \delta)D_0 \left(1 - \frac{C_1}{C_2}\lambda^{-\frac{N}{2}(1-3\varepsilon)}\right), \end{aligned}$$

and since $(\lambda - \delta)\left(1 - \frac{C_1}{C_2}\lambda^{-\frac{N}{2}(1-3\varepsilon)}\right) > \Lambda_-$ if N is large enough, we have $D_1 > \Lambda_- D_0$. We checked the base of induction.

Let us make the step of induction. Assume that for some k inequalities (9) hold. Let us prove that these inequalities also hold for $k+1$. We have

$$d_{k+1} = (1+2\delta)d_k + \delta D_k.$$

If $\delta^{1/2} D_k \le d_k$, then

$$\begin{aligned} d_{k+1} &\le (1+2\delta)d_k + \delta^{1/2} d_k \\ &= (1+2\delta+\delta^{1/2})d_k \\ &= (1+2\delta+\delta^{1/2})\max\{d_{k-1}, \delta^{1/2}D_{k-1}\}. \end{aligned}$$

If $\delta^{1/2} D_k > d_k$, then

$$\begin{aligned} d_{k+1} &\le (1+2\delta)\delta^{1/2} D_k + \delta D_k \\ &= \delta^{1/2} D_k (1+2\delta+\delta^{1/2}) \\ &= (1+2\delta+\delta^{1/2})\max\{d_{k-1}, \delta^{1/2}D_{k-1}\}. \end{aligned}$$

In order to estimate D_{k+1} we need more information on $\{d_i\}_{i=0}^k$.

Lemma 6.3. *Assume that $k^* \le N$ and (9) holds for all $k = 1, \dots, k^*$. Assume also that for $k = 1, 2, \dots, l-1$ we have $\delta^{1/2} D_k \le d_k$, and $\delta^{1/2} D_l > d_l$. Then $\delta^{1/2} D_k > d_k$ for all $k = l, l+1, \dots, k^*$.*

Proof. If δ is small enough, $1 + 2\delta + \delta^{1/2} < \Lambda_-$. Therefore, if $\delta^{1/2} D_l > d_l$ then

$$\begin{aligned} d_{l+1} &= (1+2\delta)d_l + \delta D_l \\ &< (1+2\delta+\delta^{1/2})\delta^{1/2} D_l \\ &< \Lambda_- \delta^{1/2} D_l \\ &\le \delta^{1/2} D_{l+1}. \end{aligned}$$

In the same way $d_{l+2} < \delta^{1/2} D_{l+2}$, and so on. □

Notice that Lemma 6.3 immediately implies the following statement.

Lemma 6.4. *Assume that $k^* \le N$ and (9) holds for all $k = 1, \dots, k^*$. If $d_k \ge \delta^{1/2} D_k$ for some $k \in \{1, 2, \dots, k^*\}$ then $d_k \le (1+2\delta+\delta^{1/2})^k$.*

Now let us estimate D_{k+1}. If $d_k < \delta^{1/2} D_k$, then

$$\begin{aligned} D_{k+1} &= (\lambda-\delta)D_k \left(1 - \frac{C_1 b_k d_k}{(\lambda-\delta)D_k}\right) \\ &\ge (\lambda-\delta)D_k \left(1 - \left(\frac{C_1}{\lambda-\delta}\right)\frac{\delta^{1/2}}{(\lambda-\delta)^{N-k}}\right) \\ &> (\lambda-\delta)D_k(1 - C_1\delta^{1/2}) \\ &> \Lambda_- D_k \end{aligned}$$

if δ is small enough.

If $d_k \geq \delta^{1/2} D_k$, then due to Lemma 6.4 we have $d_k \leq (1+2\delta+\delta^{1/2})^k$, and hence

$$\begin{aligned}
D_{k+1} &= (\lambda-\delta)D_k\left(1-\frac{C_1 b_k d_k}{(\lambda-\delta)D_k}\right)\\
&\geq (\lambda-\delta)D_k\left(1-\left(\frac{C_1}{\lambda-\delta}\right)\frac{(1+2\delta+\delta^{1/2})^k}{\Lambda_-^k D_0(\lambda-\delta)^{N-k}}\right)\\
&\geq (\lambda-\delta)D_k\left(1-\left(\frac{C_1}{C_2(\lambda-\delta)}\right)\frac{(1+2\delta+\delta^{1/2})^k(\lambda+\delta)^{N/2}}{\Lambda_-^k(\lambda-\delta)^{N-k}}\right)\\
&\geq (\lambda-\delta)D_k\left(1-\left(\frac{C_1}{C_2(\lambda-\delta)}\right)\frac{(1+2\delta+\delta^{1/2})^k\Lambda_+^{N/2}}{\Lambda_-^k\Lambda_-^{N-k}}\right)\\
&\geq (\lambda-\delta)D_k\left(1-\left(\frac{C_1}{C_2(\lambda-\delta)}\right)\frac{(1+2\delta+\delta^{1/2})^k\lambda^{\frac{N}{2}(1+\varepsilon)}}{\lambda^{N(1-\varepsilon)}}\right)\\
&\geq (\lambda-\delta)D_k\left(1-\left(\frac{C_1}{C_2(\lambda-\delta)}\right)(1+2\delta+\delta^{1/2})^k\lambda^{-\frac{N}{2}(1-3\varepsilon)}\right)\\
&> \Lambda_- D_k
\end{aligned}$$

if N is large enough. This concludes the proof of Lemma 6.1. □

Lemma 6.5. *Given $C_1 > 0, C_2 > 0, \lambda > 1$, $\varepsilon \in (0, \frac{1}{4})$, there exists $\delta_0 = \delta_0(C_1, C_2, \lambda, \varepsilon)$ and $N_0 \in \mathbb{N}, N_0 = N_0(C_1, C_2, \lambda, \varepsilon, \delta_0)$ such that for any $\delta \in (0, \delta_0)$ and any $N \geq N_0$ the following holds. Assume that the sequences $\{a_k\}_{k=0}^N$ and $\{A_k\}_{k=0}^N$ have the following properties:*

$$\begin{cases} a_0 = 1 \\ A_0 \geq C_2\sqrt{\widetilde{b}_0} > 0, \end{cases} \tag{10}$$

and for $k = 0, 2, \ldots, N-1$

$$\begin{cases} a_{k+1} \leq (1+2\delta)a_k + \delta A_k, \\ A_{k+1} \geq (\lambda-\delta)A_k - C_1\widetilde{b}_k a_k, \end{cases} \tag{11}$$

where the sequence $\widetilde{b}_k$ has the properties:

$$0 < \widetilde{b}_0 < \widetilde{b}_1 < \ldots < \widetilde{b}_{N-1} < 1 \leq \widetilde{b}_N,$$

and for all $k = 0, 2, \ldots, N-1$

$$(\lambda-\delta)\widetilde{b}_k \leq \widetilde{b}_{k+1} \leq (\lambda+\delta)\widetilde{b}_k.$$

Then

$$\begin{cases} a_N \leq 2\delta^{1/2}A_N \\ A_N \geq \lambda^{N(1-\varepsilon)}A_0 > \lambda^{\frac{N}{2}(1-4\varepsilon)}. \end{cases}$$

Proof. Consider the sequences $\{d_k\}$ and $\{D_k\}$ defined by (7) and (8), where we take $D_0 = A_0$. If we prove that for all $k = 0, 1, \ldots, N$

$$\begin{cases} A_k \geq D_k \\ \frac{A_k}{a_k} \geq \frac{D_k}{d_k}, \end{cases} \tag{12}$$

then the required inequalities follow from Lemma 6.1.

We will prove (12) by induction. The base of induction ($k = 0$) is provided by our choice of D_0. Let us make a step of induction. If (12) holds for some k, then

$$\begin{aligned}\frac{A_{k+1}}{a_{k+1}} &\geq \frac{(\lambda-\delta)A_k - C_1\widetilde{b}_k a_k}{(1+2\delta)a_k + \delta A_k}\\ &= (\lambda-\delta)\delta^{-1} - \frac{(1+2\delta)(\lambda-\delta)\delta^{-1} + C_1\widetilde{b}_k}{(1+2\delta)+\delta\frac{A_k}{a_k}}\\ &\geq (\lambda-\delta)\delta^{-1} - \frac{(1+2\delta)(\lambda-\delta)\delta^{-1} + C_1 b_k}{(1+2\delta)+\delta\frac{D_k}{d_k}}\\ &= \frac{D_{k+1}}{d_{k+1}}.\end{aligned}$$

Also,

$$\begin{aligned}A_{k+1} &\geq (\lambda-\delta)A_k - C_1 b_k a_k\\ &= A_k\left((\lambda-\delta) - C_1\widetilde{b}_k\frac{a_k}{A_k}\right)\\ &\geq D_k\left((\lambda-\delta) - C_1 b_k\frac{d_k}{D_k}\right)\\ &= D_{k+1}.\end{aligned}$$

Lemma 6.5 is proved. □

6.2. **Expansion of Vectors From Large Cones.** Finally, we come to the

Proof of Proposition 1. Take $\mathbf{v} \in K_p$. If $|\mathbf{v}_z| > |\mathbf{v}_{xy}|$, then the required inequalities follows just from the condition (iii). Therefore we can assume that $|\mathbf{v}_{xy}| \geq |\mathbf{v}_z|$, and normalize the vector $\mathbf{v}$ assuming that $|\mathbf{v}_{xy}| = 1$.

Denote

$$Df(p) = \begin{pmatrix}\nu(p) & m_1(p) & t_1(p)\\ m_2(p) & e(p) & t_2(p)\\ s_1(p) & s_2(p) & \lambda(p)\end{pmatrix}.$$

Given a vector $\mathbf{v} \in K_p$, denote $\mathbf{v}' = Df(p)(\mathbf{v})$, $\mathbf{v}' = \mathbf{v}'_{xy} + \mathbf{v}'_z$, $\mathbf{v}'_{xy} = \mathbf{v}'_x + \mathbf{v}'_y$. We have

$$Df(p)(\mathbf{v}) = \begin{pmatrix}\nu(p) & m_1(p) & t_1(p)\\ m_2(p) & e(p) & t_2(p)\\ s_1(p) & s_2(p) & \lambda(p)\end{pmatrix}\begin{pmatrix}\mathbf{v}_x\\ \mathbf{v}_y\\ \mathbf{v}_z\end{pmatrix} = \begin{pmatrix}\nu(p)\mathbf{v}_x + m_1(p)\mathbf{v}_y + t_1(p)\mathbf{v}_z\\ m_2(p)\mathbf{v}_x + e(p)\mathbf{v}_y + t_2(p)\mathbf{v}_z\\ s_1(p)\mathbf{v}_x + s_2(p)\mathbf{v}_y + \lambda(p)\mathbf{v}_z\end{pmatrix}$$

Conditions $(i) - (iii)$ of Proposition 1 implies that $|\nu(p)| \leq \lambda^{-1} + \delta$, $|m_1(p)|, |m_2(p)|, |t_1(p)|, |t_2(p)| \leq \delta$, $|\lambda(p)| \geq \lambda - \delta$. Also, if p belongs to the plane $\{z = 0\}$ then $s_1(p) = s_2(p) = 0$. Since $\|f\|_{C^2} \leq C_1$, for arbitrary p we have $|s_1(p)|, |s_2(p)| \leq C_1 z_p$.

Let us use the norm $\|\mathbf{v}\| = |\mathbf{v}_x| + |\mathbf{v}_y| + |\mathbf{v}_z|$. Then we have

$$\|\mathbf{v}'_{xy}\| \leq (1+2\delta)\|\mathbf{v}_{xy}\| + \delta\|\mathbf{v}_z\|$$

$$\|\mathbf{v}'_z\| \geq (\lambda-\delta)\|\mathbf{v}_z\| - C_1 z_p\|\mathbf{v}_{xy}\|$$

Denote by $\widetilde{b}_k$ a z-coordinate of $f^k(p)$, by A_k the z-component of the vector $Df^k(p)(\mathbf{v})$, and by a_k the xy-component of $Df^k(p)(\mathbf{v})$. Applying Lemma 6.5 we get (3), and (4) follows from Remark 6.2.

Now if $A_0 = |\mathbf{v}_z| \geq \eta$ then (applying Lemma 6.1 for $D_0 = \eta$) from (12) and (9) we have

$$|Df^k(p)(\mathbf{v})| \geq A_k \geq D_k \geq \Lambda_-^k D_0 \geq \lambda^{\frac{k}{2}(1-\varepsilon)}\eta > \frac{1}{2}\lambda^{\frac{k}{2}(1-\varepsilon)}\eta|\mathbf{v}| > C\lambda^{\frac{k}{2}(1-4\varepsilon)}|\mathbf{v}|$$

for every $k = 1, 2, \ldots, N$, where $C = \frac{1}{2}\eta$. Proposition 1 is proved. □

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