ELECTRONIC RESEARCH ANNOUNCEMENTS
IN MATHEMATICAL SCIENCES
Volume 16, Pages 23–29 (May 7, 2009)
S 1935-9179 AIMS (2009)
doi:10.3934/era.2009.16.23

THE SPECTRUM OF THE WEAKLY COUPLED FIBONACCI HAMILTONIAN

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(Communicated by Boris Hasselblatt)

ABSTRACT. We consider the spectrum of the Fibonacci Hamiltonian for small values of the coupling constant. It is known that this set is a Cantor set of zero Lebesgue measure. Here we study the limit, as the value of the coupling constant approaches zero, of its thickness and its Hausdorff dimension. We announce the following results and explain some key ideas that go into their proofs. The thickness tends to infinity and, consequently, the Hausdorff dimension of the spectrum tends to one. Moreover, the length of every gap tends to zero linearly. Finally, for sufficiently small coupling, the sum of the spectrum with itself is an interval. This last result provides a rigorous explanation of a phenomenon for the Fibonacci square lattice discovered numerically by Even-Dar Mandel and Lifshitz.

1. Introduction

The Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $\ell^2(\mathbb{Z})$,

$$[H_{V,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)\psi(n),$$

where $V > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$, and $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

This operator family has been studied in many papers since the early 1980's and numerous fundamental results are known. Let us recall some of them and refer the reader to the articles [7, 8, 32] for a survey of the mathematical literature and [18, 23] for the foundational physics papers on this model.

The spectrum is easily seen to be independent of ω and may therefore be denoted by Σ_V. That is, $\sigma(H_{V,\omega}) = \Sigma_V$ for every $\omega \in \mathbb{T}$. Indeed, this follows quickly from the minimality of the irrational rotation by α and strong operator convergence. It was shown by Sütő that Σ_V has zero Lebesgue measure for every $V > 0$; see [31]. Moreover, it is compact (since it is the spectrum of a bounded operator) and perfect (because the irrational rotation by α is ergodic). Thus, Σ_V is a zero-measure Cantor set. This result was recently strengthened by Cantat [5] who showed that the Hausdorff dimension of Σ_V lies strictly between zero and one.

Received by the editors January 27, 2009 and, in revised form, March 23, 2009.
2000 *Mathematics Subject Classification.* Primary: 82B44; Secondary: 37D99.
Key words and phrases. Spectrum, Fibonacci Hamiltonian, trace map, dynamically defined Cantor set, Hausdorff dimension, thickness.
D. D. was supported in part by NSF grant DMS–0800100.

Naturally, one is interested in fractal properties of Σ_V, such as its dimension, thickness, and denseness. While such a study is well-motivated from a purely mathematical perspective, we want to point out that there is significant additional interest in these quantities. In particular, it has recently been realized that the fractal dimension of the spectrum is intimately related with the long-time asymptotics of the solution to the associated time-dependent Schrödinger equation, that is, $i\partial_t \phi = H_{V,\omega}\phi$; see [9].

Fractal properties of Σ_V are by now well understood for large values of V. Work of Casdagli [6] and Sütő [30] shows that for $V \ge 16$, Σ_V is a dynamically defined Cantor set. It follows from this result that the Hausdorff dimension and the upper and lower box counting dimension of Σ_V all coincide; let us denote this common value by $\dim \Sigma_V$. Using this result, Damanik, Embree, Gorodetski, and Tcheremchantsev have shown upper and lower bounds for the dimension; see [9]. A particular consequence of these bounds is the identification of the asymptotic behavior of the dimension as V tends to infinity:

$$\lim_{V \to \infty} \dim \Sigma_V \cdot \log V = \log(1 + \sqrt{2}).$$

The paper [9] also discusses some of the implications for the dynamics of the Schrödinger equation; let us mention [11, 12] for further recent advances in this direction for the strongly coupled Fibonacci Hamiltonian.

By contrast, hardly anything about Σ_V (beyond it having Hausdorff dimension strictly between zero and one) is known for small values of V. The largeness of V enters the proofs of the existing results in critical ways. Consequently, these proofs indeed break down once the largeness assumption is dropped. The purpose of this note is to announce results about Σ_V for V sufficiently small, which are shown by completely different methods. We will indicate some of the main ideas that are used to prove these results, but we defer full details to a future publication.

We would like to emphasize that quantitative properties of regular Cantor sets such as thickness and denseness are widely used in dynamical systems (see [20, 21], [24], [19]), found an application in number theory (see [16]), but to the best of our knowledge, these kinds of techniques have never been used before in the context of mathematical physics.

2. Statement of the Results

In this section we describe our results for small coupling V. Clearly, as V approaches zero, $H_{V,\omega}$ approaches the free Schrödinger operator

$$[H_0\psi](n) = \psi(n+1) + \psi(n-1),$$

which is a well-studied object whose spectral properties are completely understood. In particular, the spectrum of H_0 is given by the interval $[-2, 2]$. It is natural to ask which spectral features of $H_{V,\omega}$ approach those of H_0. It follows from Sütő's result that the Lebesgue measure of the spectrum does not extend continuously to the case $V = 0$. Given this situation, one would at least hope that the dimension of the spectrum is continuous at $V = 0$.

It was shown by us in [10] (and independently by Cantat [5]) that Σ_V is a dynamically defined Cantor set for $V > 0$ sufficiently small (i.e., the small coupling counterpart to Casdagli's result at large coupling). A consequence of this is the equality of Hausdorff dimension and upper and lower box counting dimensions of

Σ_V in this coupling constant regime. Our first result shows that the dimension of the spectrum indeed extends continuously to $V = 0$.

Theorem 1. *We have*

$$\lim_{V \to 0} \dim \Sigma_V = 1.$$

More precisely, there are constants $C_1, C_2 > 0$ such that

$$1 - C_1 V \le \dim \Sigma_V \le 1 - C_2 V$$

for $V > 0$ sufficiently small.

Theorem 1 is a consequence of a connection between the Hausdorff dimension of a Cantor set and its denseness and thickness, along with estimates for the latter quantities. Since these notions and connections may be less familiar to at least a part of our intended audience, let us recall the definitions and some of the main results; an excellent general reference in this context is [24].

Let $C \subset \mathbb{R}$ be a Cantor set and denote by I its convex hull. Any connected component of $I \backslash C$ is called a *gap* of C. A *presentation* of C is given by an ordering $\mathcal{U} = \{U_n\}_{n \ge 1}$ of the gaps of C. If $u \in C$ is a boundary point of a gap U of C, we denote by K the connected component of $I \backslash (U_1 \cup U_2 \cup \ldots \cup U_n)$ (with n chosen so that $U_n = U$) that contains u and write

$$\tau(C, \mathcal{U}, u) = \frac{|K|}{|U|}.$$

With this notation, the *thickness* $\tau(C)$ and the *denseness* $\theta(C)$ of C are given by

$$\tau(C) = \sup_{\mathcal{U}} \inf_{u} \tau(C, \mathcal{U}, u), \qquad \theta(C) = \inf_{\mathcal{U}} \sup_{u} \tau(C, \mathcal{U}, u),$$

and they are related to the Hausdorff dimension of C by the following inequalities (cf. [24, Section 4.2]),

$$\dim_{\mathrm{H}} C \ge \frac{\log 2}{\log(2 + \frac{1}{\tau(C)})}, \qquad \dim_{\mathrm{H}} C \le \frac{\log 2}{\log(2 + \frac{1}{\theta(C)})}.$$

Due to these inequalities, Theorem 1 is a consequence of the following result:

Theorem 2. *We have*

$$\lim_{V \to 0} \tau(\Sigma_V) = \infty.$$

More precisely, there are constants $C_3, C_4 > 0$ such that

$$C_3 V^{-1} \le \tau(\Sigma_V) \le \theta(\Sigma_V) \le C_4 V^{-1}$$

for $V > 0$ sufficiently small.

Bovier and Ghez described in their 1995 paper [4] the then-state of the art concerning mathematically rigorous results for Schrödinger operators in $\ell^2(\mathbb{Z})$ with potentials generated by primitive substitutions. The Fibonacci Hamiltonian belongs to this class; more precisely, it is in many ways the most important example within this class of models. One of the more spectacular discoveries is that, in this class of models, the spectrum jumps from being an interval for coupling $V = 0$ to being a zero-measure Cantor set for coupling $V > 0$. That is, as the potential is turned on, a dense set of gaps opens immediately (and the complement of these gaps has zero Lebesgue measure). It is natural to ask about the size of these gaps, which can in fact be parametrized by a canonical countable set of gap labels; see

[3]. For some examples, these gap openings were studied in [1] and [2]. However, for the important Fibonacci case, the problem remained open. In fact, Bovier and Ghez write on p. 2321 of [4]: *It is a quite perplexing feature that even in the simplest case of all, the golden Fibonacci sequence, the opening of the gaps at small coupling is not known!*

There is a perturbative approach to this problem for a class of models that includes the Fibonacci Hamiltonian by Sire and Mosseri; see [27] and [22, 26, 28, 29] for related work. While their work is non-rigorous, it gives quite convincing arguments in favor of linear gap opening; see especially [27, Section 5]. It would be interesting to make their approach mathematically rigorous.

Our next result resolves this issue completely and shows that, in the Fibonacci case, all gaps indeed open linearly:

Theorem 3. *For $V > 0$ sufficiently small, the boundary points of a gap in the spectrum Σ_V depend smoothly on the coupling constant V. Moreover, given any such one-parameter family $\{U_V\}_{V>0}$, where U_V is a gap of Σ_V and the boundary points of U_V depend smoothly on V, we have that*

$$\lim_{V\to 0} \frac{|U_V|}{V}$$

exists and belongs to $(0, \infty)$.

Our next result concerns the sum set $\Sigma_V + \Sigma_V = \{E_1 + E_2 : E_1, E_2 \in \Sigma_V\}$. This set is equal to the spectrum of the so-called square Fibonacci Hamiltonian. Here, one considers the Schrödinger operator

$$\begin{aligned}[H_V^{(2)}\psi](m,n) =&\psi(m+1,n) + \psi(m-1,n) + \psi(m,n+1) + \psi(m,n-1) + \\ &+ V\left(\chi_{[1-\alpha,1)}(m\alpha \mod 1) + \chi_{[1-\alpha,1)}(n\alpha \mod 1)\right)\psi(m,n)\end{aligned}$$

in $\ell^2(\mathbb{Z}^2)$. The theory of tensor products of Hilbert spaces and operators then implies that $\sigma(H_V^{(2)}) = \Sigma_V + \Sigma_V$. This operator and its spectrum have been studied numerically and heuristically by Even-Dar Mandel and Lifshitz in a series of papers [13, 14, 15]. A different but similar model was studied by Sire [26]. Their study suggested that at small coupling, the spectrum of $\Sigma_V + \Sigma_V$ is not a Cantor set; quite on the contrary, it has no gaps at all.

Our final theorem confirms this observation:

Theorem 4. *For $V > 0$ sufficiently small, we have that $\Sigma_V + \Sigma_V$ is an interval.*

Notice that Theorem 4 is a direct consequence of Theorem 2 and the famous Gap Lemma, which was used by Newhouse to construct persistent tangencies and generic diffeomorphisms with an infinite number of attractors (the so-called "Newhouse phenomenon"):

Gap Lemma (Newhouse [20])**.** *If $C_1, C_2 \subset \mathbb{R}^1$ are Cantor sets such that*

$$\tau(C_1) \cdot \tau(C_2) > 1,$$

then either one of these sets is contained entirely in a gap of the other set, or $C_1 \cap C_2 \neq \emptyset$.

3. Comments on the Proofs

We will exploit the following relation between the spectrum of the Fibonacci Hamiltonian and the dynamical system known as the trace map. By the trace map we mean the following polynomial map:

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \quad T(x, y, z) = (2xy - z, x, y).$$

The trace map preserves the following quantity (called the Fricke-Vogt invariant):

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1.$$

We denote by S_V the surface $\left\{I(x, y, z) = \frac{V^2}{4}\right\}$ invariant under T.

The relation between the trace map and the spectrum of the Fibonacci Hamiltonian is given by the following statement.

Theorem 5 (Sütő [30])**.** *An energy $E \in \mathbb{R}$ belongs to the spectrum Σ_V of $H_{V,\omega}$ if and only if the positive semiorbit of the point $(\frac{E-V}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.*

Denote by ℓ_V the line

$$\ell_V = \left\{ \left(\frac{E - V}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}.$$

It is easy to check that $\ell_V \subset S_V$. It is known that the set

$$\Lambda_V = \{p \in S_V : \ \mathcal{O}_T(p) \text{ is } \ \text{bounded}\}$$

is a hyperbolic set for all $V \neq 0$, see [6], [10], [5]. Moreover, its stable manifolds are transversal to ℓ_V for small V. Therefore the set $W^s(\Lambda_V) \cap \ell_V$ is affinely equivalent to Σ_V. In particular, Σ_V is a dynamically defined Cantor set for small values of V.

The surface S_0 is the so-called Cayley cubic; it has four conic singularities and can be represented as a union of a two dimensional sphere (with four conic singularities) and four unbounded components. The restriction of the trace map to the sphere is a pseudo-Anosov map (a factor of a hyperbolic map of a two-torus), and its Markov partition can be presented explicitly (see [6] or [10]). For small values of V, the map $T : S_V \to S_V$ "inherits" the hyperbolicity of this pseudo-Anosov map everywhere away from singularities. The dynamics near the singularities needs to be considered separately. Due to the symmetries of the trace map, it is enough to consider the dynamics of T near one of the singularities, say, near the point $p = (1, 1, 1)$. The set $Per_2(T)$ of periodic orbits of period two is a smooth curve that contains the point p and intersects S_V at two points (denote them by p_1 and p_2) for $V > 0$. Also, this curve is a normally hyperbolic manifold, and we can use the normally hyperbolic theory (see [17], [25]) to study the behavior of T in a small neighborhood of p; see [10] for details. In particular, since the distance between p_1 and p_2 is of order V, and the gaps in the spectrum are formed by the points of intersection of $W^{ss}(p_1)$ and $W^{ss}(p_2)$ with the line ℓ_V, the size of a given gap is of order V as $V \to 0$, which implies Theorem 3.

In order to estimate the thickness (and the denseness) of the spectrum Σ_V, we notice first that the Markov partition for $T : S_0 \to S_0$ can be continuously extended to a Markov partition for $T : \Lambda_V \to \Lambda_V$, and while the size of the elements of these Markov partitions remains bounded, the size of the distance between them is of order V. The natural approach now is to use the distortion property (see, e.g., [24]) to show that for the iterated Markov partition, the ratio of the distance between the elements to the size of an element is of the same order. The main technical problem here is again the dynamics of the trace map near the singularities, since the curvature of S_V is very large there for small V. Nevertheless, one can still estimate the distortion that is obtained during a transition through a neighborhood of a singularity and prove boundedness of the distortion for arbitrarily large iterates of the trace map. This implies Theorem 2.

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