

## THE SPECTRUM AND THE SPECTRAL TYPE OF THE OFF-DIAGONAL FIBONACCI OPERATOR

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**ABSTRACT.** We consider Jacobi matrices with zero diagonal and off-diagonals given by elements of the hull of the Fibonacci sequence and show that the spectrum has zero Lebesgue measure and all spectral measures are purely singular continuous. In addition, if the two hopping parameters are distinct but sufficiently close to each other, we show that the spectrum is a dynamically defined Cantor set, which has a variety of consequences for its local and global fractal dimension.

**1. Introduction.** The study of the Fibonacci operator was initiated in the early 1980's by Kohmoto et al. [17] and Ostlund et al. [23]. At that point in time, the interest in this model was based mainly on the existence of an exact renormalization group procedure and the appearance of critical eigenstates and zero measure Cantor spectrum. Only shortly thereafter, Shechtman et al. [28] reported their discovery of structures, now called quasicrystals, whose diffraction exhibits sharp Bragg peaks with rotational symmetries that ensure the aperiodicity of the structure in question.

The Fibonacci sequence is the central model of a quasicrystal in one dimension. Indeed, it is aperiodic and pure point diffractive and, moreover, it belongs to virtually all classes of mathematical models of quasicrystals that have since been proposed. We refer to the reader to [1] for a relatively recent account of the mathematics related to the modelling and study of quasicrystals.

Thus, the study of the Fibonacci operator was further motivated by the interest in electronic spectra and transport properties of one-dimensional quasicrystals. Consequently, apart from the almost Mathieu operator, the Fibonacci operator has been the most heavily studied quasi-periodic operator in the last three decades; compare the survey articles [5, 6, 30].

Partly due to the choice of the model in the foundational papers [17, 23], the mathematical literature on the Fibonacci operator has so far only considered the diagonal model, that is, a discrete one-dimensional Schrödinger operator with potential given by the Fibonacci sequence. Given the connection to quasicrystals and hence aperiodic point sets, and particularly cut-and-project sets, it is however equally (if not more) natural to study the off-diagonal model. That is, one considers a Jacobi matrix with zero diagonal and off-diagonals given by the Fibonacci sequence. This would correspond to hopping over unequal distances, the sites being points on the real line generated by the cut-and-project scheme that produces the Fibonacci point set. While we will not explain these concepts here in detail, we do

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refer the reader to [1] for background and we will describe the resulting model carefully in the next section. For further motivation to study the off-diagonal model, we mention that it has been the object of interest in a number of physics papers; see, for example, [13, 14, 18, 27, 31, 32].

In fact, our interest in the off-diagonal model was triggered by the numerical results for separable higher-dimensional models obtained recently by Even-Dar Mandel and Lifshitz in [13, 14]. We intend to explore the observations reported in these papers and prove some of the phenomena rigorously, and for this task, we will need the basic spectral results for the off-diagonal Fibonacci operator obtained in this paper.

The paper is organized as follows. In Section 2 we describe in detail the model we study and the results we obtain for it. The results concerning the spectrum as a set are proved in Section 3 and the results concerning the spectral type are proved in Section 4.

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**2. Model and Results.** In this section we describe the off-diagonal Fibonacci operator and the results for it that are proved in subsequent sections. Let  $a, b$  be two positive real numbers and consider the Fibonacci substitution,

$$S(a) = ab, \quad S(b) = a.$$

This substitution rule extends to finite and one-sided infinite words by concatenation. For example,  $S(aba) = abaab$ . Since  $S(a)$  begins with  $a$ , one obtains a one-sided infinite sequence that is invariant under  $S$  by iterating the substitution rule on  $a$  and taking a limit. Indeed, we have

$$S^k(a) = S^{k-1}(S(a)) = S^{k-1}(ab) = S^{k-1}(a)S^{k-1}(b) = S^{k-1}(a)S^{k-2}(a). \quad (1)$$

In particular,  $S^k(a)$  starts with  $S^{k-1}(a)$  and hence there is a unique one-sided infinite sequence  $u$ , the so-called Fibonacci substitution sequence, that starts with  $S^k(a)$  for every  $k$ . The hull  $\Omega_{a,b}$  is then obtained by considering all two-sided infinite sequences that locally look like  $u$ ,

$$\Omega_{a,b} = \{\omega \in \{a,b\}^{\mathbb{Z}} : \text{every subword of } \omega \text{ is a subword of } u\}.$$

It is known that, conversely, every subword of  $u$  is a subword of every  $\omega \in \Omega_{a,b}$ . In this sense,  $u$  and all elements of the hull  $\omega$  look exactly the same locally.

We wish to single out a special element of  $\Omega_{a,b}$ . Notice that  $ba$  occurs in  $u$  and that  $S^2(a) = aba$  begins with  $a$  and  $S^2(b) = ab$  ends with  $b$ . Thus, iterating  $S^2$  on  $b|a$ , where  $|$  denotes the eventual origin, we obtain as a limit a two-sided infinite sequence which belongs to  $\Omega_{a,b}$  and coincides with  $u$  to the right of the origin. This element of  $\Omega_{a,b}$  will be denoted by  $\omega_s$ .

Each  $\omega \in \Omega_{a,b}$  generates a Jacobi matrix  $H_\omega$  acting in  $\ell^2(\mathbb{Z})$ ,

$$(H_\omega \psi)_n = \omega_{n+1} \psi_{n+1} + \omega_n \psi_{n-1}.$$

With respect to the standard orthonormal basis  $\{\delta_n\}_{n \in \mathbb{Z}}$  of  $\ell^2(\mathbb{Z})$ , where  $\delta_n$  is one at  $n$  and vanishes otherwise, this operator has the matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & 0 & \omega_{-1} & 0 & & \\ \ddots & \omega_{-1} & 0 & \omega_0 & 0 & \\ & 0 & \omega_0 & 0 & \omega_1 & \ddots \\ & & 0 & \omega_1 & 0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

and it is clearly self-adjoint.

This family of operators,  $\{H_\omega\}_{\omega \in \Omega_{a,b}}$ , is called the off-diagonal Fibonacci model. Of course, the structure of the Fibonacci sequence disappears when  $a = b$ . In this case, the hull consists of a single element, the constant two-sided infinite sequence taking the value  $a = b$ , and the spectrum and the spectral measures of the associated operator  $H_\omega$  are well understood. For this reason, we will below always assume that  $a \neq b$ . Nevertheless, the limiting case, where we fix  $a$ , say, and let  $b$  tend to  $a$  is of definite interest.

Our first result concerns the spectrum of  $H_\omega$ . For  $S \subset \mathbb{R}$ , we denote by  $\dim_H S$  the Hausdorff dimension of  $S$  and by  $\dim_B S$  the box counting dimension of  $S$  (which is then implicitly claimed to exist); see [8] or [24] for the definition of these fractal dimensions.

**Theorem 1.** *Suppose  $a, b > 0$  and  $a \neq b$ . Then, there exists a compact set  $\Sigma_{a,b} \subset \mathbb{R}$  such that  $\sigma(H_\omega) = \Sigma_{a,b}$  for every  $\omega \in \Omega_{a,b}$ , and*

- (i)  $\Sigma_{a,b}$  has zero Lebesgue measure.
- (ii) The Hausdorff dimension  $\dim_H \Sigma_{a,b}$  is an analytic function of  $a$  and  $b$ .
- (iii)  $0 < \dim_H \Sigma_{a,b} < 1$ .

Also, there exists  $\varepsilon_0 > 0$  such that if  $\frac{a^2+b^2}{2ab} < 1 + \varepsilon_0$  (in other words, if  $a$  and  $b$  are close enough), then

- (iv) The spectrum  $\Sigma_{a,b}$  is a Cantor set that depends continuously on  $a$  and  $b$  in the Hausdorff metric.
- (v) For every small  $\delta > 0$  and every  $E \in \Sigma_{a,b}$ , we have

$$\begin{aligned} \dim_H((E - \delta, E + \delta) \cap \Sigma_{a,b}) &= \dim_B((E - \delta, E + \delta) \cap \Sigma_{a,b}) \\ &= \dim_H \Sigma_{a,b} = \dim_B \Sigma_{a,b}. \end{aligned}$$

- (vi) Denote  $\alpha = \dim_H \Sigma_{a,b}$ , then the Hausdorff  $\alpha$ -measure of  $\Sigma_{a,b}$  is positive and finite.

Properties (ii)–(vi) are related to hyperbolicity of the trace map described in the next section, see [3, 4, 9]. It is reasonable to expect that in fact the properties (iv)–(vi) for  $\Sigma_{a,b}$  hold for all positive  $a \neq b$ , but one of the steps in the proof is currently missing as will be explained in Section 3.

Next, we turn to the spectral type of  $H_\omega$ .

**Theorem 2.** *Suppose  $a, b > 0$  and  $a \neq b$ . Then, for every  $\omega \in \Omega_{a,b}$ ,  $H_\omega$  has purely singular continuous spectrum.*

Throughout the rest of the paper we will only consider  $a, b > 0$  with  $a \neq b$ . Theorem 1 is proved in Section 3 and Theorem 2 is proved in Section 4.

**3. The Trace Map and its Relation to the Spectrum.** The spectral properties of  $H_\omega$  are closely related to the behavior of the solutions to the difference equation

$$\omega_{n+1}u_{n+1} + \omega_n u_{n-1} = Eu_n. \quad (2)$$

Denote

$$U_n = \begin{pmatrix} u_n \\ \omega_n u_{n-1} \end{pmatrix}.$$

Then  $u$  solves (2) (for every  $n \in \mathbb{Z}$ ) if and only if  $U$  solves

$$U_n = T_\omega(n, E)U_{n-1}, \quad (3)$$

(for every  $n \in \mathbb{Z}$ ), where

$$T_\omega(n, E) = \frac{1}{\omega_n} \begin{pmatrix} E & -1 \\ \omega_n^2 & 0 \end{pmatrix}.$$

Note that  $\det T_\omega(n, E) = 1$ . Iterating (3), we find

$$U_n = M_\omega(n, E)U_0,$$

where

$$M_\omega(n, E) = T_\omega(n, E) \times \cdots \times T_\omega(1, E).$$

With the Fibonacci numbers  $\{F_k\}$ , generated by  $F_0 = F_1 = 1$ ,  $F_{k+1} = F_k + F_{k-1}$  for  $k \geq 1$ , we define

$$x_k = x_k(E) = \frac{1}{2} \text{Tr} M_{\omega_s}(F_k, E).$$

For example, we have

$$\begin{aligned} M_{\omega_s}(F_1, E) &= \frac{1}{a} \begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix} \\ M_{\omega_s}(F_2, E) &= \frac{1}{b} \begin{pmatrix} E & -1 \\ b^2 & 0 \end{pmatrix} \frac{1}{a} \begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix} = \frac{1}{ab} \begin{pmatrix} E^2 - a^2 & -E \\ Eb^2 & -b^2 \end{pmatrix} \\ M_{\omega_s}(F_3, E) &= \frac{1}{a} \begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix} \frac{1}{b} \begin{pmatrix} E & -1 \\ b^2 & 0 \end{pmatrix} \frac{1}{a} \begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix} = \frac{1}{a^2 b} \begin{pmatrix} E^3 - Ea^2 - Eb^2 & -E^2 + b^2 \\ E^2 a^2 - a^4 & -Ea^2 \end{pmatrix} \end{aligned}$$

and hence

$$x_1 = \frac{E}{2a}, \quad x_2 = \frac{E^2 - a^2 - b^2}{2ab}, \quad x_3 = \frac{E^3 - 2Ea^2 - Eb^2}{2a^2 b}. \quad (4)$$

**Lemma 1.** *We have*

$$x_{k+1} = 2x_k x_{k-1} - x_{k-2} \quad (5)$$

for  $k \geq 2$ . Moreover, the quantity

$$I_k = x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_k x_{k-1} - 1 \quad (6)$$

is independent of both  $k$  and  $E$  and it is given by

$$I = \frac{(a^2 + b^2)^2}{4a^2 b^2} - 1.$$

*Proof.* Since  $\omega_s$  restricted to  $\{n \geq 1\}$  coincides with  $u$  and the prefixes  $s_k$  of  $u$  of length  $F_k$  obey  $s_{k+1} = s_k s_{k-1}$  for  $k \geq 2$  by construction (cf. (1)), the recursion (5) follows as in the diagonal case; compare [5, 6, 29]. This recursion in turn implies readily that (6) is  $k$ -independent.

In particular, the  $x_k$ 's are again generated by the trace map

$$T(x, y, z) = (2xy - z, x, y)$$

and the preserved quantity is again

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1.$$

The only difference between the diagonal and the off-diagonal model can be found in the initial conditions. How are  $x_1, x_0, x_{-1}$  obtained? Observe that the trace map is invertible and hence we can apply its inverse twice to the already defined quantity  $(x_3, x_2, x_1)$ . We have

$$T^{-1}(x, y, z) = (y, z, 2yz - x)$$

and hence, using (4),

$$\begin{aligned} (x_1, x_0, x_{-1}) &= T^{-2}(x_3, x_2, x_1) \\ &= T^{-2}\left(\frac{E^3 - 2Ea^2 - Eb^2}{2a^2b}, \frac{E^2 - a^2 - b^2}{2ab}, \frac{E}{2a}\right) \\ &= T^{-1}\left(\frac{E^2 - a^2 - b^2}{2ab}, \frac{E}{2a}, 2\frac{(E^2 - a^2 - b^2)E}{4a^2b} - \frac{E^3 - 2Ea^2 - Eb^2}{2a^2b}\right) \\ &= T^{-1}\left(\frac{E^2 - a^2 - b^2}{2ab}, \frac{E}{2a}, \frac{E}{2b}\right) \\ &= \left(\frac{E}{2a}, \frac{E}{2b}, 2\frac{E^2}{4ab} - \frac{E^2 - a^2 - b^2}{2ab}\right) \\ &= \left(\frac{E}{2a}, \frac{E}{2b}, \frac{a^2 + b^2}{2ab}\right) \end{aligned}$$

It follows that

$$\begin{aligned} I(x_{k+1}, x_k, x_{k-1}) &= I(x_1, x_0, x_{-1}) \\ &= \frac{E^2}{4a^2} + \frac{E^2}{4b^2} + \frac{(a^2 + b^2)^2}{4a^2b^2} - 2\frac{E^2(a^2 + b^2)}{8a^2b^2} - 1 \\ &= \frac{(a^2 + b^2)^2}{4a^2b^2} - 1 \end{aligned}$$

for every  $k \geq 0$ . □

It is of crucial importance for the spectral analysis that, as in the diagonal case, the invariant is energy-independent and strictly positive when  $a \neq b$ !

**Lemma 2.** *The spectrum of  $H_\omega$  is independent of  $\omega$  and may be denoted by  $\Sigma_{a,b}$ . With*

$$\sigma_k = \{E \in \mathbb{R} : |x_k| \leq 1\},$$

*we have*

$$\Sigma_{a,b} = \bigcap_{k \geq 1} \sigma_k \cup \sigma_{k+1}. \quad (7)$$

Moreover, for every  $E \in \Sigma_{a,b}$  and  $k \geq 2$ ,

$$|x_k| \leq 1 + \left( \frac{(a^2 + b^2)^2}{4a^2b^2} - 1 \right)^{1/2} \quad (8)$$

and for  $E \notin \Sigma_{a,b}$ ,  $|x_k|$  diverges super-exponentially.

*Proof.* It is well known that the hull  $\Omega_{a,b}$  together with the standard shift transformation is minimal. In particular, every  $\omega \in \Omega_{a,b}$  may be approximated pointwise by a sequence of shifts of any other  $\tilde{\omega} \in \Omega_{a,b}$ . The associated operators then converge strongly and we get  $\sigma(H_\omega) \subseteq \sigma(H_{\tilde{\omega}})$ . Reversing the roles of  $\omega$  and  $\tilde{\omega}$ , the first claim follows.

So let  $\Sigma_{a,b}$  denote the common spectrum of the operators  $H_\omega$ ,  $\omega \in \Omega_{a,b}$ . We have  $\|H_\omega\| \leq \max\{2a, 2b\}$ . Thus,  $\Sigma_{a,b} \subseteq [-\max\{2a, 2b\}, \max\{2a, 2b\}] =: I_{a,b}$ . For  $E \in I_{a,b}$ , we have that at least one of  $x_1, x_0$  belongs to  $[-1, 1]$ . This observation allows us to use the exact same arguments Sütő used to prove (7) for the diagonal model in [29].

The only point where care needs to be taken is the claim that  $\sigma_k$  is the spectrum of the periodic Jacobi matrix obtained by repeating the values  $\omega_s$  takes on  $\{1 \leq n \leq F_k\}$  periodically on the off-diagonals. This, however, follows from the general theory of periodic Jacobi matrices, which relies on the diagonalization of the monodromy matrix (which is  $M_{\omega_s}(F_k, E)$  in this case) in order to obtain Floquet solutions and in particular discriminate between those energies that permit exponentially growing solutions and those that do not. This distinction works just as well here, but one needs to use that the  $\omega_n$ 's that enter in the  $U_n$ 's are uniformly bounded away from zero and infinity.

Thus, after paying attention to this fact, we may now proceed along the lines of Sütő. Let us describe the main steps of the argument. Since at least one of  $x_1, x_0$  belongs to  $[-1, 1]$ , we have a result analogous to [29, Lemma 2] with the same proof as given there. Namely, the sequence  $\{x_k\}_{k \geq 0}$  is unbounded if and only if there exists  $k$  such that  $|x_k| > 1$  and  $|x_{k+1}| > 1$ . Moreover, we then have  $|x_{k+l}| > c^{Fl}$  for some  $c > 1$  and all  $l \geq 0$ . This shows

$$\sigma_k \cup \sigma_{k+1} = \bigcup_{l \geq 0} \sigma_{k+l}.$$

Using now the fact that the  $F_k$  periodic Jacobi matrices with spectrum  $\sigma_k$  converge strongly to  $H_{\omega_s}$ , we obtain

$$\Sigma_{a,b} \subseteq \bigcap_{k \geq 1} \overline{\bigcup_{l \geq 0} \sigma_{k+l}} = \bigcap_{k \geq 1} \overline{\sigma_k \cup \sigma_{k+1}} = \bigcap_{k \geq 1} \sigma_k \cup \sigma_{k+1},$$

since the spectra  $\sigma_k$  and  $\sigma_{k+1}$  are closed sets. Thus, we have one inclusion in (7).

Next, suppose  $E \in \bigcap_{k \geq 1} \sigma_k \cup \sigma_{k+1}$ . If  $k \geq 1$  is such that  $|x_k| > 1$ , then  $|x_{k-1}| \leq 1$  and  $|x_{k+1}| \leq 1$ . Since we have

$$x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 = \frac{(a^2 + b^2)^2}{4a^2b^2} - 1,$$

this implies

$$x_k = x_{k+1}x_{k-1} \pm \left( 1 - x_{k+1}^2 - x_{k-1}^2 + x_{k+1}^2x_{k-1}^2 + \frac{(a^2 + b^2)^2}{4a^2b^2} - 1 \right)^{1/2}$$

and hence

$$|x_k| \leq |x_{k+1}x_{k-1}| + \left( (1 - x_{k+1}^2)(1 - x_{k-1}^2) + \left( \frac{(a^2 + b^2)^2}{4a^2b^2} - 1 \right) \right)^{1/2}$$

which, using  $|x_{k-1}| \leq 1$  and  $|x_{k+1}| \leq 1$  again, implies the estimate (8) for  $E \in \bigcap_{k \geq 1} \sigma_k \cup \sigma_{k+1}$ . We will show in the next section that the boundedness of the sequence  $\{x_k\}_{k \geq 0}$  implies that, for arbitrary  $\omega \in \Omega_{a,b}$ , no solution of the difference equation (2) is square-summable at  $+\infty$ . Consequently, such  $E$ 's belong to  $\Sigma_{a,b}$ .<sup>1</sup> This shows the other inclusion in (7) and hence establishes it. Moreover, it follows that (8) holds for every  $E \in \Sigma_{a,b}$ .

Finally, from the representation (7) of  $\Sigma_{a,b}$  and our observation above about unbounded sequences  $\{x_k\}_{k \geq 0}$ , we find that  $|x_k|$  diverges super-exponentially for  $E \notin \Sigma_{a,b}$ . This concludes the proof of the lemma.  $\square$

**Lemma 3.** *For every  $E \in \mathbb{R}$ , there is  $\gamma(E) \geq 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_\omega(n, E)\| = \gamma(E),$$

*uniformly in  $\omega \in \Omega_{a,b}$ .*

*Proof.* This follows directly from the uniform subadditive ergodic theorem; compare [11, 12, 15, 20].  $\square$

**Lemma 4.** *The set  $\mathcal{Z}_{a,b} := \{E \in \mathbb{R} : \gamma(E) = 0\}$  has zero Lebesgue measure.*

*Proof.* This is one of the central results of Kotani theory; see [19] and also [7]. Note that these papers only discuss the diagonal model. Kotani theory for Jacobi matrices is discussed in Carmona-Lacroix [2] and the result needed can be deduced from what is presented there. For a recent reference that states a result sufficient for our purpose explicitly, see Remling [26].  $\square$

**Lemma 5.** *We have  $\Sigma_{a,b} = \mathcal{Z}_{a,b}$ .*

*Proof.* The inclusion  $\Sigma_{a,b} \supseteq \mathcal{Z}_{a,b}$  holds by general principles. For example, one can construct Weyl sequences by truncation when  $\gamma(E) = 0$ . The inclusion  $\Sigma_{a,b} \subseteq \mathcal{Z}_{a,b}$  can be proved in two ways. Either one uses the boundedness of  $x_k$  for energies  $E \in \Sigma_{a,b}$  to prove explicit polynomial upper bounds for  $\|M_\omega(n, E)\|$  (as in [16] for  $\omega = \omega_s$  or in [11] for general  $\omega \in \Omega_{a,b}$ ), or one combines the proof of the absence of decaying solutions at  $+\infty$  for  $E \in \Sigma_{a,b}$  given in the next section with Osceledec's Theorem, which states that  $\gamma(E) > 0$  would imply the existence of an exponentially decaying solution at  $+\infty$ . Here we use one more time that  $U_n$  is comparable in norm to  $(u_n, u_{n-1})^T$ .  $\square$

*Proof of Theorem 1.* The existence of the uniform spectrum  $\Sigma_{a,b}$  was shown in Lemma 3 and the fact that  $\Sigma_{a,b}$  has zero Lebesgue measure follows from Lemmas 4 and 5. The set of bounded orbits of the restriction of the trace map  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  to the invariant surface  $I(x, y, z) = C \equiv \frac{(a^2 + b^2)^2}{4a^2b^2} - 1, C > 0$ , is hyperbolic; see [3] (and also [9] for  $C$  sufficiently small and [4] for  $C$  sufficiently large). Due to Lemma 2,

<sup>1</sup>This follows by a standard argument: If  $E \notin \Sigma_{a,b}$ , then  $(H_\omega - E)^{-1}$  exists and hence  $(H_\omega - E)^{-1}\delta_0$  is an  $\ell^2(\mathbb{Z})$  vector that solves (2) away from the origin. Choosing its values for  $n \geq 1$ , say, and then using (2) to extend it to all of  $\mathbb{Z}$ , we obtain a solution that is square-summable at  $+\infty$ .

the points of the spectrum correspond to the points of the intersection of the line of the initial conditions

$$l_{a,b} \equiv \left\{ \left( \frac{E}{2a}, \frac{E}{2b}, \frac{a^2 + b^2}{2ab} \right) : E \in \mathbb{R} \right\}$$

with the stable manifolds of the hyperbolic set of bounded orbits. Properties (ii) and (iii) can be proved in exactly the same way as Theorem 6.5 in [3]. The line  $l_{a,b}$  intersects the stable lamination of the hyperbolic set transversally for sufficiently small  $C > 0$ , as can be shown in the same way as for the diagonal Fibonacci Hamiltonian with a small coupling constant; see [9]. Therefore the spectrum  $\Sigma_{a,b}$  for close enough  $a$  and  $b$  is a dynamically defined Cantor set, and the properties (iv)–(vi) follow; see [8, 9, 21, 22, 25, 24] and references therein.  $\square$

Notice that a proof of the transversality of the line  $l_{a,b}$  to the stable lamination of the hyperbolic set of bounded orbits for arbitrary  $a \neq b$  would imply the properties (iv)–(vi) for these values of  $a$  and  $b$ .

**4. Singular Continuous Spectrum.** In this section we prove Theorem 2. Given the results from the previous section, we can follow the proofs from the diagonal case quite closely.

*Proof of Theorem 2.* Since the absence of absolutely continuous spectrum follows from zero measure spectrum, we only need to show the absence of point spectrum. It was shown by Damanik and Lenz [10] that, given any  $\omega \in \Omega_{a,b}$  and  $k \geq 1$ , the restriction of  $\omega$  to  $\{n \geq 1\}$  begins with a square

$$\omega_1 \dots \omega_{2F_k} \dots = \omega_1 \dots \omega_{F_k} \omega_1 \dots \omega_{F_k} \dots$$

such that  $\omega_1 \dots \omega_{F_k}$  is a cyclic permutation of  $S^k(a)$ . By cyclic invariance of the trace, it follows that  $\text{Tr} M_\omega(F_k, E) = 2x_k(E)$  for every  $E$ .

The Cayley-Hamilton Theorem, applied to  $M_\omega(F_k, E)$ , says that

$$M_\omega(F_k, E)^2 - (\text{Tr} M_\omega(F_k, E)) M_\omega(F_k, E) + I = 0,$$

which, by the observations above, translates to

$$M_\omega(2F_k, E) - 2x_k M_\omega(F_k, E) + I = 0.$$

If  $E \in \Sigma_{a,b}$  and  $u$  is a solution of the difference equation (2), it therefore follows that

$$U(2F_k + 1) - 2x_k U(F_k + 1) + U(1) = 0.$$

If  $u$  does not vanish identically, this shows that  $u_n \not\rightarrow 0$  as  $n \rightarrow \infty$  since the  $x_k$ 's are bounded above and the  $\omega_n$ 's are bounded below away from zero. In particular, if  $E \in \Sigma_{a,b}$ , then no non-trivial solution of (2) is square-summable at  $+\infty$  and hence  $E$  is not an eigenvalue. It follows that the point spectrum of  $H_\omega$  is empty.  $\square$

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