DYNAMICAL SYSTEMS

PROBLEMS

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(1) Which of the following maps are topologically transitive (minimal, topologically mixing)?
- identity map on a circle;
- irrational rotation of a circle;
- expanding endomorphisms of a circle;
- hyperbolic automorphism of a torus;
- topological Bernoulli shift.

Definition 0.0.1. A product of a dynamical systems $f_1 : M_1 \to M_1$ and $f_2 : M_2 \to M_2$ is a map $F : M_1 \times M_2 \to M_1 \times M_2$, $F(x, y) = (f_1(x), f_2(y)), (x, y) \in M_1 \times M_2$.

(2) Is the product of two topologically transitive (minimal, topologically mixing) systems topologically transitive (minimal, topologically mixing)?

(3) Let α be irrational and $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the homeomorphism of the 2-torus given by $f(x, y) = (x + \alpha, x + y)$. Prove that f is topologically transitive.

(4) Prove that f from the previous problem is minimal.

(5) Is it possible to construct a transitive homeomorphism of S^2 that has exactly one dense orbit? Only countable number of points with dense orbits?

(6) Show that the topological entropy of an isometry is zero.

(7) Let X, Y be compact metric spaces, $f : X \to X$ and $g : Y \to Y$ be homeomorphisms. Assume that there exists a continuous

map (onto) $h : X \to Y$ such that $h \circ f = g \circ h$. Show that $h_{top}(f) \geq h_{top}(g)$.

(8) Denote by $P_n(\sigma)$ the number of periodic points of (not necessarily minimal) period n for a topological Bernoulli shift $\sigma : \Sigma_m \to \Sigma_m$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \log P_n(\sigma) = h_{top}(\sigma).$$

Check if this is true for an expending endomorphism $E_m : S^1 \to S^1$.

(9) The same question as in the previous problem, for a hyperbolic automorphism of a torus, $A : \mathbb{T}^2 \to \mathbb{T}^2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(10) Construct an example of a topologically transitive diffeomorphism of a two dimensional torus for which the only minimal set is a fixed point.

(11) Find $\Omega(f)$, $C(f)$, and Milnor's attractor of f.

Consider $F : \mathbb{T}^2 \to \mathbb{T}^2$, $\mathbb{T}^2 = S^1 \times S^1$, $F = f \times f$, and $G : \mathbb{T}^2 \to \mathbb{T}^2$, $G = f \times R_\alpha$, where R_α is an irrational rotation of S^1.

(12) Find $\Omega(F)$, $C(F)$, and Milnor's attractor of F. The same for G.

(13) Is the rotation of a circle $R_\alpha : S^1 \to S^1$ structurally stable?

(14) Prove that $E_2 : S^1 \to S^1$ has an invariant subset homeomorphic to a Cantor set.

(15) Is it possible to find a topological conjugacy between $\sigma_3 : \Sigma_3 \to \Sigma_3$ and $\sigma_2 : \Sigma_2 \to \Sigma_2$? Semiconjugacy?

(16) Let p be a saddle fixed point of a diffeomorphism $f : M \to M$. Let x be a homoclinic point, i.e. $x \in W^s_f(p) \cap W^u_f(p)$. Prove that x is non-wandering point but not recurrent.

Set $D = \{(x, y) |\, x^2 + y^2 < 1\}$ and consider a map $f : D \times S^1 \to D \times S^1$, $f(x, y, \phi) = (\frac{x}{10}, \frac{y}{10}, \phi)$. Let $g : D \times S^1 \to D \times S^1$ be C^1-close to f.

(17) Prove that maximal attractor of g is homeomorphic to a circle.

(18) Prove that maximal attractor of g is a smooth closed curve.

Set $B = \{(x, y) |\, |x| < 1, |y| < 1\}$ and consider a map $h : B \times S^1 \to B \times S^1$, $h(x, y, \phi) = (10x, \frac{y}{10}, \phi)$. Let $k : B \times S^1 \to B \times S^1$ be C^1-close to h.

(19) Prove that $\cap_{n \in \mathbb{Z}} k^n(B \times S^1)$ is homeomorphic to a circle.

(20) Prove that $\cap_{n \in \mathbb{Z}} k^n(B \times S^1)$ is a smooth closed curve.

(21) Given a measure-preserving transformation f in a finite measure space (M, μ) and a measurable subset $A \subset M$ of positive measure, define the *conditional measure* μ_A by

$$\mu_A(B) := \frac{\mu(B \cap A)}{\mu(A)}$$

The first return map $f_A : A \to A$ is defined by $f_A(x) = f^k(x)$, where $k \in \mathbb{N}$ is the smallest natural number for which $f^k(x) \in A$. Prove that $f_A : (A, \mu_A) \to (A, \mu_A)$ is a measure-preserving transformation.

Remark. The first return map is often called derivative map, or Poincare map.

(22) Is it true that powers of an ergodic (mixing) measure-preserving map are ergodic (mixing)?

(23) Give an example of a continuous map of the real line that does not have non-trivial finite invariant Borel measures.

(24) Describe all the ergodic measures of the homeomorphism of the torus $f : \mathbb{T}^2 \to \mathbb{T}^2$, $f(x, y) = (x, x + y) (\bmod 1)$.

(25) Consider the following map $\alpha : \Sigma_2^+ \to \Sigma_2^+$,

$$(\alpha(\omega))_i = \begin{cases} 1 - \omega_i, & if\ \omega_i = 1\ for\ \ all\ \ j < i, \\ \omega_i, & otherwise. \end{cases}$$

Prove that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is uniquely ergodic and describe the invariant measure.

(26) Construct an example of a homeomorphism of a compact metric space with infinite topological entropy.

(27) Calculate the metric entropy of the tent map (with respect to the Lebesgue measure).

(28) Calculate the metric entropy of baker's transformation (with respect to the Lebesgue measure).

(29) Suppose a measure-preserving transformation $f : (M, \mu) \to (M, \mu)$ has a generator with k elements. Prove that $h_\mu(f) \leq \log k$.

(30) Construct an example of a homeomorphism f of a compact metric space with finite topological entropy that does not have a measure of maximal entropy (an invariant measure μ is called "a measure of maximal entropy" if $h_\mu(f) = h_{top}(f)$).

(31) Find the box counting dimension of the set

$$E = \{0, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\} \subset \mathbb{R}.$$

(32) Construct an example of a Cantor set with large (close to 1) box counting dimension but small (close to 0) thickness (use Moran's formula).

(33) Give an example of a hyperbolic set Λ such that periodic points are not dense in Λ.

(34) Let Λ_i be a hyperbolic set of $f_i : U_i \to M_i$, $i = 1, 2$. Prove that $\Lambda_1 \times \Lambda_2$ is a hyperbolic set of $f_1 \times f_2 : U_1 \times U_2 \to M_1 \times M_2$.

(35) Prove that any contracting C^1-diffeomorphism of $\mathbb{R}$ is topologically conjugated to a linear contraction.

Consider expending maps

$$E_2 : S^1 \to S^1,\ E_2(x) = 2x(\text{mod}1), \quad \text{and}$$

$$E_3 : S^1 \to S^1,\ E_3(x) = 3x(\text{mod}1).$$

Denote by F the product map, $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F = E_2 \times E_3$.

(36) Is F minimal? Transitive? Topologically mixing?

(37) Find $P_n(F)$ - the number of periodic points of (not necessarily minimal) period n.

(38) Find $h_{top}(F)$.

(39) Prove that there exists a semiconjugacy between a topological Bernoulli shift $\sigma : \Sigma_6^+ \to \Sigma_6^+$ and $F : \mathbb{T}^2 \to \mathbb{T}^2$.

(40) Is it possible to find a point $x \in \mathbb{T}$ such that $\omega(x)$ is homeomorphic to a circle? To a Cantor set? To a product of a circle and a Cantor set?

Consider the map $F_{\alpha,\beta} : \Sigma^2 \times S^1 \to \Sigma^2 \times S^1$, $\omega \in \Sigma^2$, $\varphi \in S^1$,

$$F_{\alpha,\beta}(\omega, \varphi) = \begin{cases} (\sigma(\omega), R_\alpha(\varphi)), & \text{if } \omega_0 = 0; \\ (\sigma(\omega), R_\beta(\varphi)), & \text{if } \omega_0 = 1. \end{cases}$$

(41) For which pairs (α, β) the map $F_{\alpha,\beta}$ is transitive?

Consider the map $g : [-1, 1] \times [-1, 1] \times [-1, 1] \to \mathbb{R}^3$,

$$g(x, y, z) = \begin{cases} (\frac{x}{100} + \frac{1}{2}, 100y - 10, 100z), & \text{if } y \in [0, 1]; \\ (\frac{x}{100} - \frac{1}{2}, 100y + 10, 100z), & \text{if } y \in [-1, 0). \end{cases}$$

Let Λ be the set of points such that all iterates of g (positive and negative) are defined.

(42) Describe Λ. Find $h_{\text{top}}(g|_\Lambda)$.

(43) What is $\dim_H \Lambda$?

Consider the map $f : S^1 \to S^1$,

$$f(x) = \begin{cases} 4x \pmod 1), & \text{if } x \in [0, \frac{1}{2}); \\ 6x \pmod 1), & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

Let μ be Lebesgue measure on S^1.

(44) Show that f is an expending map and preserve μ.

(45) Is f topologically conjugate to a linear expending map?

(46) Is $f : (S^1, \mu) \to (S^1, \mu)$ ergodic? Is it mixing?

(47) Calculate $h_{\text{top}}(f)$ and $h_\mu(f)$.

(48) Give an example of a homeomorphism $f : M \to M$ of a compact metric space such that

$$d(f^n(x), f^n(y)) \to 0 \quad as \quad n \to \infty$$

for every pair $x, y \in M$.

(49) Prove the following. Let $f : M \to M$ be a C^1-map of a complete Riemannian manifold to itself. Then f is a contracting map if and only if the norm of the differential is bounded by a constant $\lambda < 1$.

Definition 0.0.2. A topological dynamical system is called minimal if every orbit is dense.

(50) Prove that if α is irrational then the rotation R_α is minimal.

(51) Prove that the decimal expansion of the number 2^n may begin with any finite number of digits.

(52) Show that the set of points with dense orbits under E_2 has Lebesgue measure 1.

(53) Is the solenoid pathwise-connected?

(54) Produce some computer graphics of the Julia set of the following maps:

a) $z \mapsto z^2 + c$ for several different $c \in \mathbb{C}$;

b) $z \mapsto z^2 \frac{2z^2-1}{2-z^3}$; $z \mapsto e^{i\pi\sqrt{5}} z^2 \frac{z-4}{4z-1}$;

c) $z \mapsto z - z^2 + \frac{z^3}{z_0}$, $z_0 = -0.41 + 0.54i$;

d) $z \mapsto z - z^2$, $z \mapsto z - z^4$, $z \mapsto i(z + z^2)$;

e) $z \mapsto e^{i\pi\sqrt{5}z} + iz^2$;

f) $z \mapsto \frac{-iz^2}{1+z^2}$;

g) $z \mapsto \frac{1}{10}e^z$.

What properties of Fatou and Julia sets did you use to generate these pictures? Can you explain (or guess) what kind of Fatou components one can see there?

(55) Show that the two polynomials $z + z^2$ and $z + z^2 + z^3$ are topologically, but not analytically, nor formally, conjugate near the origin.

(56) Let $R(z) = \frac{z}{2-z^2}$. Show that $F(R)$ has an attracting component of infinite connectivity.

(57) Prove that if rational functions R and S are permutable ($RS \equiv SR$), then any periodic Fatou component of R is also a periodic Fatou component of S, and vice versa.

(58) Prove that the Julia set of a Blaschke product is either S^1 or a Cantor set contained in S^1.

Consider the logistic family of functions

$$F_\lambda(x) = \lambda x(1 - x).$$

(59) For which values of λ does F_λ have an attracting fixed point at $x = 0$? For which values of λ does F_λ have a nonzero attracting fixed point?

(60) Describe the bifurcation that occurs when $\lambda = 1$. Sketch the bifurcation diagram near $\lambda = 1$.

(61) Describe the bifurcation that occurs when $\lambda = 3$. Sketch the bifurcation diagram near $\lambda = 3$.

(62) Describe the bifurcation that occurs when $\lambda = -1$. Sketch the bifurcation diagram near $\lambda = -1$.

(63) Compute the Schwarzian Derivative of the function $M(x) = \frac{ax+b}{cx+d}$. Prove that $S(M \circ f) = Sf$ for every smooth function f.

(64) Sketch the phase curves of the vector field

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

(65) Sketch the phase curves of the vector field

$$\begin{cases} \dot{x} = \quad y + (x^2 + y^2)(1 - x^2 - y^2) \\ \dot{y} = -x + (x^2 + y^2)(1 - x^2 - y^2) \end{cases}$$

What is $\omega(p)$, if $p = (\frac{1}{2}, \frac{1}{2})$? If $p = (0, 0)$? If $p = (10, 10)$?

(66) Sketch the set

$$A = \{z \in \mathbb{C} \mid |z| \le 1 \text{ or } |\sqrt{2}z - 1 - i| \le 1 \text{ or } |\sqrt{2}z - 1 + i| \le 1 \text{ or } |\sqrt{2}z + 1 - i| \le 1 \text{ or } |\sqrt{2}z + 1 + i| \le 1\}.$$

(67) Find all the fixed points for each of the following complex functions and determine whether they are attracting, repelling, or neutral.
a) $Q_2(z) = z^2 + 2$
b) $F(z) = z^2 + z + 1$
c) $F(z) = iz^2$
d) $F(z) = -\frac{1}{z}$
e) $F(z) = 2z(i - z)$
f) $F(z) = -iz(1 - z)/2$
g) $F(z) = z^3 + (i + 1)z$

(68) Show that $z_0 = e^{2\pi i/7}$ is a periodic point of period 3 for the map $Q_0(z) = z^2$. Is this periodic orbit attracting, repelling, or neutral?