Ergodic Theory, Final

This is a take-home midterm. Solve these problems and turn it in either in person or via email on or before June 13, 2018. You can use any textbooks, notes, etc., but not direct help of other people.

Problem 1.

Let f be a homeomorphism of the compact metric space M.

a) Show that the union of supports of all invariant Borel probability measures of f is a closed set;

b) Prove that support of any invariant Borel probability measure of f is a subset of $\Omega(f)$ (here $\Omega(f)$ is the non-wandering set of f);

c) Prove that $h_{top}(f) = h_{top}\left(f|_{\Omega(f)}\right)$;

d) Is it true that the union of supports of all invariant Borel probability measures of f is exactly $\Omega(f)$? Prove or give a counterexample.

Problem 2.

Define the map $T : [0,1] \to [0,1]$ by $T(x) = 4x(1-x)$. Define the measure μ by

$$\mu(B) = \frac{1}{\pi} \int_B \frac{1}{\sqrt{x(1-x)}} dx.$$

a) Check that μ is a probability measure;

b) Show that T preserves μ;

c) Prove that $T : ([0,1], \mu) \to ([0,1], \mu)$ is ergodic;

d) Prove that $T : ([0,1], \mu) \to ([0,1], \mu)$ is mixing;

e) Show that $h_\mu(T) = \log 2$;

f) Find $h_{top}(T)$;

g) Find all the eigenvalues of the corresponding Koopman operator.

Problem 3.

Consider the map $F_{\alpha,\beta} : \Sigma^2 \times S^1 \to \Sigma^2 \times S^1$, $\omega \in \Sigma^2$, $\varphi \in S^1$,

$$F_{\alpha,\beta}(\omega, \varphi) = \begin{cases} (\sigma(\omega), R_\alpha(\varphi)), & \text{if } \omega_0 = 0; \\ (\sigma(\omega), R_\beta(\varphi)), & \text{if } \omega_0 = 1. \end{cases}$$

a) For which pairs (α, β) the map $F_{\alpha,\beta}$ is transitive?

b) Find $h_{top}(F_{\alpha,\beta})$;

c) Let μ be $(1/2, 1/2)$ - Bernoulli measure on Σ^2, and $\nu = \mu \times Leb_{S^1}$. Check that ν is invariant measure for $F_{\alpha,\beta}$;

d) TRUE or FALSE: If for some pair (α, β) the measure ν is ergodic for $F_{\alpha,\beta}$, then it is also mixing.