ERGODIC THEORY, HW # 1

Homework will not be collected or graded. Nevertheless, please, make sure that you understand how to solve the problems. Interaction between students is strongly encouraged.

Problem 1.

Prove that if $f : M \to M$ preserves the measure μ then, given any $k \geq 2$, the iterate f^k also preserves μ. Is the converse true?

Problem 2.

Prove that for any finite sequence of digits there exists a power of 2 that starts with this sequence.

Problem 3.

Let $F : (M, \mu) \to (M, \mu)$ and $G : (N, \nu) \to (N, \nu)$ be measure preserving maps. How would you define $f \times g : M \times N \to M \times N$? Prove that it preserve the measure $\mu \times \nu$. Is it true that if both F and G are ergodic, then $F \times G$ is also ergodic?

Problem 4.

Consider expending maps

$$E_2 : S^1 \to S^1,\ E_2(x) = 2x(\text{mod}1), \quad \text{and}$$

$$E_3 : S^1 \to S^1,\ E_3(x) = 3x(\text{mod}1).$$

Denote by F the product map, $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F = E_2 \times E_3$. Is it possible to find a point $x \in \mathbb{T}$ such that $\omega(x)$ is homeomorphic to a circle? To a Cantor set? To a product of a circle and a Cantor set?

Problem 5.

Define the map $T : [0, 1] \to [0, 1]$ by $T(x) = 4x(1 - x)$. Define the measure μ by

$$\mu(B) = \frac{1}{\pi} \int_B \frac{1}{\sqrt{x(1-x)}} dx.$$

a) Check that μ is a probability measure;

b) Show that T preserves μ.