ERGODIC THEORY, HW # 2

Homework will not be collected or graded. Nevertheless, please, make sure that you understand how to solve the problems. Interaction between students is strongly encouraged.

Problem 1.

Let $M = [0,1]$, and $f : M \to M$ be the function defined by

$$\begin{cases} 2x, & \text{if } 0 \le x < \frac{1}{3}; \\ 2x - 2/3, & \text{if } \frac{1}{3} \le x < \frac{1}{2}; \\ 2x - 1/3, & \text{if } \frac{1}{2} \le x < \frac{2}{3}; \\ 2x - 1, & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

Show that f is ergodic with respect to the Lebesgue measure on $[0,1]$.

Problem 2.

Let M be a compact metric space, $f : M \to M$ be a homeomorphism, and μ be an invariant ergodic Borel probability measure. Prove that

$$\text{supp}\, \mu = \{x \in M \mid \text{any neighborhood of } x \text{ has positive measure}\}$$

is an invariant set.

Problem 3.

Let M be a compact metric space, $f : M \to M$ be a homeomorphism, and μ be an invariant ergodic Borel probability measure. Prove that the positive semi-orbit $\{f^n(x) \mid n \ge 0\}$ of μ-almost every point $x \in M$ is dense in the support of μ.

Problem 4.

Suppose $f : M \to M$ has two different ergodic invariant probability measures μ and ν (defined on the same σ-algebra of subsets of M). Prove that there are subsets $A, B \subset M$ such that $\mu(A) = \nu(B) = 1$, $\mu(B) = \nu(A) = 0$. In other words, two different ergodic measures must be singular with respect to each other.

Problem 5.

Prove that for Lebegue almost every $x \in (0,1)$, if x is represented as a decimal fraction $x = 0.x_1x_2x_3\ldots$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_j = 4.5$$